The Calabi Ansatz

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1 Introduction

Calabi-Yau metrics, with holonomy contained in $SU(n)$, form one of five families of special geometries within Berger's classification of the Riemannian holonomy groups of simply-connected, irreducible, non-symmetric[1] Riemannian manifolds [3]. They also have a special role to play within algebraic geometry, and mathematical physics, in Euclidean theories of gravity and string theory.

Examples of Kähler metrics, with holonomy contained in $U(n)$, are relatively easy to construct: one could take any holomorphic submanifold of $\mathbb{CP}^n$ with the Fubini-Study metric, or more generally any holomorphic submanifold of any Kähler manifold, see also [4], [5]. The explicit construction of Calabi-Yau metrics is more difficult, despite Yau's proof of the Calabi conjecture, showing the existence of Ricci-flat Kähler metrics in a given Kähler class, given constraints of topology and complex geometry.

In these notes, we study a construction of Calabi [6], later rediscovered in the physics literature by Page-Pope [7], of complete Calabi-Yau metrics on the total space of the canonical line bundle over a Kähler-Einstein metric.

1.1 Kähler and Calabi Yau metrics

Throughout this introduction, let (M, g) be a complete, connected, Riemannian manifold with $\dim(M) = 2m$. We begin by recalling the following definitions:

Definition 1.1. Let (M, g, J) be a Riemannian manifold with a complex structure J. We say that (M, g, J) is **Kähler**, if the associated **Kähler form** $\omega \in \Omega^{1,1}(M)$, defined by $\omega(u, v) := g(Jv, w)$ for $u, v \in \Gamma(TM)$, satisfies:

$$d\omega = 0 \tag{1}$$

Since Kähler manifolds have an (almost) complex structure, they are automatically orientable, so we may define a volume form $dV \in \Omega^{2m}(M)$ with respect to g, i.e. a nowhere-vanishing trivialising section of this bundle. Since J acts by orthogonal transformations, the m^{th} exterior power of the Kähler form is then a constant multiple of this volume form. The convention we follow is that:

$$\omega^m = m!\, dV$$

Example 1.1. On $\mathbb{C}^m$ with the standard complex structure, and standard Hermitian metric induced by the Euclidean metric, i.e. $g = \sum_j dz_j \otimes d\bar{z}_j$, we have the Kähler form given by:

$$\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j = \sum_j dx_j \wedge dy_j$$

where we identify $(z_1, \ldots, z_m) \in \mathbb{C}^m$ with $(x_1, y_1, \ldots, x_m, y_m) \in \mathbb{R}^{2m}$. The standard volume form is:

$$dV = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_m \wedge dy_m$$

A fact, which we will need later, is that the volume form dV may also be used to define a *point-wise* hermitian structure on $(p, m-p)$-forms, $0 \leq p \leq m$:

$$\Omega^{p,m-p}(M) \times \Omega^{p,m-p}(M) \longrightarrow \mathbb{C} \qquad\qquad (\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle$$

This is defined by:

$$\alpha \wedge \bar{\beta} = i^{p^2} \langle \alpha, \beta \rangle dV$$

[1] Note that Calabi-Yau metrics are necessarily Ricci-flat [1]. Since symmetric spaces are complete, and homogeneous [2, P.236], in particular if they are Ricci-flat, they flat.

In particular for $s \in \Omega^{m,0}(M)$:

$$m!\, s \wedge \bar{s} = i^{m^2} \langle s, s \rangle \omega^m \tag{2}$$

We will now briefly digress into some holomorphic vector bundle theory, to understand the general context of our construction [4, Prop 2.6.23]:

Lemma 1.1. *If $E \to M$ is a holomorphic vector bundle, we may naturally extend the derivative $\bar{\partial}$ on p,q-forms to a $\mathbb{C}$-linear operator $\bar{\partial}_E : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$, such that for all $\alpha \in \Omega^{p,q}(E)$, $f \in C^\infty(M)$:*

$$\bar{\partial}_E^2(\alpha) = 0 \qquad\qquad \bar{\partial}_E(f\alpha) = \bar{\partial} f \wedge \alpha + f \bar{\partial}_E \alpha$$

Now for any connection $\nabla : \Omega^0(E) \to \Omega^1(E)$ on E, consider the splitting: $\nabla = \nabla^{1,0} \oplus \nabla^{0,1} : \Omega^0(E) \longrightarrow \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$. We say the connection on E is **compatible with the holomorphic structure** if $\nabla^{0,1} = \bar{\partial}_E$ on sections of E, i.e. on $\Omega^{0,0}(E) = \Omega^0(E)$. If this connection is also compatible with some hermitian structure on this bundle, then ∇ is the **Chern connection** for this hermitian structure: one always exists, and it is furthermore unique.

An example of this Chern connection can be found on the complex line bundle $L := \bigwedge^{m,0}(T^*M) \to M$. Given a connection ∇ on L, and open set $U \subseteq M$, we can find a local non-vanishing section s, so that we can write:

$$\nabla(s) = \psi \otimes s \tag{3}$$

for some connection 1-form $\psi \in C^\infty(U, T^*U_{\mathbb{C}})$.

To obtain a Chern connection, we use some of additional structure on L: it is the underlying complex vector bundle to the **canonical bundle** $K_M \to M$, the m^{th} power of the *holomorphic* cotangent bundle [4, Prop 2.6.4]. In the case of our hermitian structure on $\Omega^{m,0}(M)$, the Chern connection is torsion-free, and coincides with the induced Levi-Civita connection of (M, g) [4, 4.A.7].

There are two formulae we will need from (3) for the Calabi ansatz. Firstly, since the Levi-Cevita connection is torsion-free:

$$ds = \partial s + \bar{\partial} s = \psi \wedge s \tag{4}$$

Second, since the Levi-Cevita connection is metric compatible:

$$\begin{aligned} d\langle s, s\rangle &= \langle \nabla(s), s\rangle + \langle s, \overline{\nabla(s)}\rangle \\ &= \langle s, s\rangle \otimes \left(\psi + \bar{\psi}\right) \Rightarrow \\ d\log\langle s, s\rangle &= 1 \otimes \left(\psi + \bar{\psi}\right) \end{aligned} \tag{5}$$

Since L is a complex line bundle, the curvature is given by $d\psi$. The curvature of this bundle relates to the curvature of the Kähler metric on M via the **Ricci form** $\rho \in \Omega^{1,1}(M)$. For $u, v \in \Gamma(TM)$, this is defined by:

$$\rho(u, v) := \mathrm{Ric}(Ju, v)$$

This form has the property that $\rho = i\mathrm{tr}_{\mathbb{C}} F_\nabla$, for ∇ the Levi-Civita connection on $\bigwedge^{1,0}(T^*M)$ with respect to the Kähler metric, [4, 4.A.11]. Since $\mathrm{tr}_{\mathbb{C}} F_\nabla$ is the curvature of the Chern connection for the induced hermitian metric on the canonical bundle $L = \det_{\mathbb{C}}(\bigwedge^{1,0}(T^*M))$, we have:

Lemma 1.2. *Let (M, g, J) be Kähler. Then $\rho = id\psi$, where $d\psi$ is the curvature of the Chern connection ψ for the induced hermitian structure on the canonical bundle $L = \det_{\mathbb{C}}(\bigwedge^{1,0}(T^*M))$.*

Proof. On any tensor product bundle $E_1 \otimes E_2$ with the natural tensor product connection $\nabla_1 \otimes \nabla_2$, the action of the curvature on sections splits as a tensor product $F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}$. Let (E, g, ∇) be $\bigwedge^{1,0}(T^*M)$, identified with the holomorphic cotangent bundle, ∇ be the Levi-Civita connection, and g be the hermitian metric with local holomorphic basis $e^1 \dots e^n$, and define $F_\nabla e^j = \sum_i e^i \otimes F_\nabla^{ij}$ where F_∇^{ij} is a $\mathfrak{u}(m)$-valued 2-form. If we consider the curvature of the complex determinant bundle $\det_{\mathbb{C}}(E)$, then the induced action of F_∇ on $e^1 \wedge \ldots \wedge e^n$ is as follows:

$$\begin{aligned} F_\nabla\left(e^1 \wedge \ldots \wedge e^n\right) &= \sum_j e^1 \wedge \ldots \wedge F_\nabla e^j \wedge \ldots \wedge e^n \\ &= \sum_{j,i} e^1 \wedge \ldots \wedge e^i \wedge \ldots \wedge e^n \otimes F_\nabla^{ij} \\ &= \sum_j e^1 \wedge \ldots \wedge e^i \wedge \ldots \wedge e^n \otimes F_\nabla^{jj} \\ &= \mathrm{tr}_{\mathbb{C}}(F_\nabla) e^1 \wedge \ldots \wedge e^n \end{aligned}$$

□

With this vector bundle theory understood, we return to a special class of Kähler manifolds:

Definition 1.2. A Kähler manifold (M, g, J) is **Kähler-Einstein** if the Ricci curvature satisfies:

$$\mathrm{Ric}\,(u, v) = \lambda g\,(u, v)$$

for some $\lambda \in C^\infty(M)$. Note that if $\dim_{\mathbb{R}} M > 2$, then we must have a constant $\lambda = \frac{\mathrm{scal}}{n}$. Also, by Lemma (1.2), an alternative restatement of this condition is:

$$id\psi = \rho = \lambda\omega$$

Let us make a few remarks about Kähler-Einstein metrics before moving on. The sign of λ can have important topological consequences for M: for example, if $\lambda > 0$, since the scalar curvature is constant, then $\mathrm{Ric} \geq \frac{\mathrm{scal}}{n} > 0$, thus we may apply Bonnet-Myers theorem. This implies that the diameter of M is bounded, so it must be compact, and furthermore it must have finite fundamental group. Much progress has been made on the classification of compact Kähler-Einstein manifolds. For example, in positive scalar curvature, $n = 4$, in [8], it is shown that the only possible Kähler-Einstein manifolds are: $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, and $\mathrm{Bl}_{p_1\ldots p_k}\mathbb{CP}^2$, for $3 \leq k \leq 8$, where $\mathrm{Bl}_{p_1\ldots p_k}\mathbb{CP}^2$ denotes the blow-up of $\mathbb{CP}^2$ along distinct isolated points $p_1 \ldots p_k$. Here, we give $\mathbb{CP}^2$ the Fubini-Study metric[2], and $\mathbb{CP}^1 \times \mathbb{CP}^1$ has the product metric. For more references on this, see [9, Ch.11].

A particularly important class of Kähler-Einstein metrics occurs when $\lambda = 0$: in this case, the metric is Ricci-flat, and the universal cover will be Calabi-Yau. We will recall some of the properties of Calabi-Yau metrics, following [1][3].

Definition 1.3. Let (M, g, J) be a Kähler manifold, and $\Omega \in \Omega^{m,0}(M)$. We say that (M, g, J, Ω) is **Calabi-Yau**, if:

$$d\Omega = 0 \tag{6}$$

$$m!\,\Omega \wedge \bar{\Omega} = i^{m^2}\omega^m \tag{7}$$

We call Ω a **holomorphic volume form** for (M, g, J). By definition, it has unit norm with respect to the hermitian metric (2) on $\Omega^{m,0}(M)$.

There are a few implications to this definition, which we will now discuss. The first point is that this definition implies that $\bigwedge^{m,0}(T^*M)$ is trivial. We can see this explicitly: our second equation gives us that Ω is non-vanishing, so any point e of the line bundle $\bigwedge^{m,0}(T^*M)$ may be written as $e = z\Omega$, where $z \in \mathbb{C}$ will parametrize the fibre coordinate of $M \times \mathbb{C}$.

The second point to make concerns the Ricci form $\rho \in \Omega^{1,1}(M)$. We have that $d\Omega = \bar{\partial}\Omega = \psi \wedge \Omega = 0$ where ψ is the induced Levi-Civita connection one-form. Since Ω is a non-vanishing $(m, 0)$-form, this implies $\mathrm{Im}\psi = 0$, and by (5), since the hermitian norm of Ω is constant, $\psi = -\bar{\psi}$. Therefore $\psi = 0$, so $\bigwedge^{m,0}(T^*M)$ is flat, and $\rho = 0$ by Lemma 1.2.

If M is simply-connected, then Ricci-flat Kähler metrics correspond to Calabi-Yau metrics in the following way: one may use the flatness condition on the canonical bundle to trivialise it by a non-zero parallel section. This section may not be globally defined when M is not simply connected, so canonical bundle may not be trivial. However, one can avoid this by working on the universal cover of M instead.

2 Calabi Construction

As discussed in the introduction, the problem that motivated the Calabi construction was to construct non-homogeneous Ricci-flat Kahler metrics, this is done on the total space of the canonical bundle over a Kähler manifold, viewed as a holomorphic line bundle. Certainly, if we want the metric to be Calabi-Yau, then we must impose that the base is at least Kähler, but we will see that there will be extra conditions on the base metric depending on the curvature of this bundle.

Let (M, J, g_M) be a Kähler manifold, and $\pi : L \to M$ be the canonical bundle with fibre $F = \mathbb{C}$. Locally at least, the tangent space TL splits, so an ansatz for a metric g on $L|_U$ for some open $U \subseteq M$ would be to split the tangent space orthogonally at $p \in L|_U$:

$$g = u(p)g_U + v(p)g_F \tag{8}$$

[2]In any dimension, $\mathbb{CP}^n$ with Fubini-Study is an example of a positive Kähler-Einstein metric.
[3]Although we drop the condition of compactness.

To make sure this is ansatz is well-defined, independent of trivialisation, we would require at least a global splitting of TL. This can be achieved by fixing a lift of TM to TL, i.e. a connection on L. The natural choice would be to use the Chern connection on L.

Now, away from the zero section of L, $L\backslash M \cong P \times (0, \infty)$, where P is the principal circle bundle as defined by the hermitian structure (2). So, we can write a global splitting $T(L\backslash M) \cong \ker d\pi \oplus H$, where H is the horizontal subspace defined by the kernel of a connection one-form on P, and $\ker d\pi \cong T\mathbb{C}$.

If we choose some (possibly non-holomorphic) local section $s \in C^\infty(U, P)$, so that as in (3):

$$\nabla(s) = \psi \otimes s$$

We know by (5) that $-\psi = \bar{\psi}$, so $i\psi$ is a real 1-one form, so on $P|_U \cong U \times S^1 \ni (x, \theta)$, define the real 1-form on P:

$$\sigma := d\theta - i\psi$$

This is a S^1-invariant one-form which is the identity on $\ker d\pi$, i.e. σ is a connection form on P. This means that if the local coordinates for $L|_U \cong U \times S^1 \times (0, \infty)$ are (x, θ, r), then the one-forms appearing in the fibre metric should be dr and $d\theta - i\psi$.

We will make the further assumption in our ansatz (8) that u, v depend only on r. Then by promoting $d\theta \mapsto \sigma$ for the flat metric $dr^2 + r^2 d\theta^2$ on $\mathbb{C}$, we can write a warped product of a metric on the fibre $\mathbb{C}$ and the metric on M as:

$$g = u(r) g_M + v(r)\left(dr^2 + r^2\sigma^2\right) \tag{9}$$

Now, for any choice of smooth functions u, v, this defines a metric on $L\backslash M$. Furthermore, if ω_M is the Kähler form on M, then a natural ansatz for the Kähler form for g on L is given by:

$$\omega = u(r)\omega_M - v(r) r dr \wedge \sigma \tag{10}$$

With this ansatz for the Kähler structure fixed, we will make some additional comments. We can write every section $\tilde{s}$ of L as zs for some $s \in C^\infty(M, P)$, $z \in C^\infty(M, \mathbb{C})$, possibly degenerately. So locally, by Leibniz rule, and equation (3), we may write the section:

$$\nabla(zs) = (dz + z\psi) \otimes s$$

For ease of notation, let us identify ψ with its pull-back via π, so that we write $(dz + z\psi)$ as a 1-form on $L|_U$. Locally, we may consider z as a coordinate on our fibre. Let us investigate what this looks in polar coordinates. Write $z = re^{i\theta}$, so that:

$$dz = e^{i\theta}(dr + ird\theta) \qquad\qquad dz + z\psi = e^{i\theta}(dr + ir(d\theta - i\psi))$$

This is a well-defined tensor on $L|_U$, and can be locally extended to the zero section at $z = 0$. To make an ansatz for the holomorphic volume form for the metric (9), we further claim that we can use the derivative of the tautological form zs, i.e.

$$\Omega = d(zs) = (dz + z\psi) \wedge s \tag{11}$$

This is well defined on L as the S^1-action on this section is trivial.

Proposition 2.1. *On some open subset of L, with Ω as defined above, and*

$$g = u(r) g_M + v(r)\left(dr^2 + r^2\sigma^2\right) \qquad\qquad \omega = u(r)\omega_M - v(r) r dr \wedge \sigma \tag{12}$$

(g, ω, Ω) is a Calabi-Yau structure iff g_M is Kähler-Einstein, and u, v satisfy:

$$v = \frac{2m}{\mathrm{rscal}(g_M)} \frac{du}{dr} \qquad\qquad 2 = u^m v$$

Proof. We must check this structure is Kähler, i.e. $d\omega = 0$, that it is furthermore Calabi-Yau, i.e. $d\Omega = 0$ and $\langle \Omega, \Omega \rangle = 1$. As for the first point:

$$\begin{aligned} d\omega &= \frac{du}{dr} dr \wedge \omega_M + v r dr \wedge d\sigma \\ &= dr \wedge \left(\frac{du}{dr}\omega_M - i v r d\psi\right) \end{aligned}$$

Then clearly, for this to vanish, it is necessary that g_M be Kähler-Einstein, as in definition (1.2): i.e. that ω_M is proportional to $id\psi$. Again, g_M is in particular Einstein, thus the ratio is constant for $m > 1$, and equal to the scalar curvature over the real dimension of M, i.e.:

$$\frac{\mathrm{scal}(g_M)}{2m}\omega_M = id\psi \tag{13}$$

Define $T := \frac{\mathrm{scal}}{2m}$. If (13) holds, then for ω to be closed, we must have the ODE in u, v:

$$v = \frac{1}{rT}\frac{du}{dr}$$

Now for the Calabi-Yau conditions. Clearly Ω is closed, but we must check unit norm:

$$\begin{gathered}(m+1)!\Omega\wedge\bar{\Omega} = i^{(m+1)^2}\langle\Omega,\Omega\rangle\omega^{(m+1)}\\ (m+1)!(dr+ir\sigma)\wedge s\wedge(dr-ir\sigma)\wedge\bar{s} = -i^{(m+1)^2}\langle\Omega,\Omega\rangle\,(m+1)\,u^m vr\omega_M^m\wedge dr\wedge\sigma\\ -(m+1)!(-1)^m 2irdr\wedge\sigma\wedge\frac{i^{m2}}{m!}\wedge\omega_M^m = -i^{(m+1)^2}\langle\Omega,\Omega\rangle(m+1)u^m vr\omega_M^m\wedge dr\wedge\sigma\Rightarrow\\ 2 = \langle\Omega,\Omega\rangle u^m v\end{gathered}$$

□

Now let us consider solving the ODE explicitly. In terms of u, we get the ODE:

$$2 = u^m\frac{1}{rT}\frac{du}{dr} \tag{14}$$

Integrating, we get general solution, for some constant C:

$$u = \left(T\,(m+1)\,r^2 + C\right)^{\frac{1}{m+1}} \qquad\qquad v = 2\left(T\,(m+1)\,r^2 + C\right)^{-\frac{m}{m+1}}$$

Re-defining our parameter r: $r \longmapsto r(m+1)^{-\frac{1}{2}}$, our metric (9) becomes:

$$g = \left(Tr^2 + C\right)^{\frac{1}{m+1}} g_M + \frac{2}{m+1}\left(Tr^2 + C\right)^{-\frac{m}{m+1}}\left(dr^2 + r^2\sigma^2\right) \tag{15}$$

This metric is the one considered by Calabi in [6], also see [10, Th.8.1]. It is Calabi-Yau, by construction, and as we will show, in the next section, it is smooth and complete on the total space of L, provided g_M is complete, and $T > 0, C > 0$. First, however, we give an explicit example in the case where $m = 1$:

Example 2.1. The **Eguchi-Hanson** metric over TS^2, with base manifold $\mathbb{CP}^1 \cong S^2$ with the Fubini-Study metric. The Fubini-Study metric on S^2 occurs as the base of a Riemannian submersion:

$$\left(S^3, ds_3^2\right) \mapsto \left(S^2, \frac{1}{4}ds_2^2\right)$$

Where ds_n^2 is the canonical metric, induced from the Euclidean metric on $\mathbb{R}^{n+1}$. We use isometry:

$$\left(S^3, ds_3^2\right) \cong \left(SU(2), \sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)$$

where σ_i is the the canonical left-invariant basis of $TSU(2)$, and the $U(1)$-action on S^3 generates a vector-field dual to σ_1. Using this form of the metric, we write the Fubini-Study metric at $T_I SU(2)$, normalised so that $T = 1$ as $\sigma_2^2 + \sigma_3^2$. So we can write the Eguchi-Hanson metric as follows:

$$g = \left(r^2 + C\right)^{\frac{1}{2}}\left(\sigma_2^2 + \sigma_3^2\right) + \frac{1}{2}\left(r^2 + C\right)^{-\frac{1}{2}}\left(dr^2 + r^2\sigma_1^2\right) \tag{16}$$

To give this in a more recognisable form, we write this using the parametrization:

$$t^2 = \left(r^2 + C\right)^{\frac{1}{2}}$$

$$g_{EH} = \left(1 - \frac{C}{t^4}\right)^{-1} dt^2 + t^2\left(\left(1 - \frac{C}{t^4}\right)\sigma_1^2 + {\sigma_2}^2 + {\sigma_3}^2\right)$$

In this parametrisation, the asymptotic behavior as $t \to \infty$ is the limit $C \to 0$, which locally gives the Euclidean metric. A Kähler form for this metric is given by:

$$\omega_{EH} = tdt \wedge \sigma_1 + t^2 \sigma_2 \wedge \sigma_3$$

Note that this agrees with our convention:

$$\omega_{EH}^2 = 2t^3 dt \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 2dV_{EH}$$

Also, the holomorphic volume form is given by:

$$\Omega_{EH} = \left(\left(1 - \frac{C}{t^4}\right)^{-\frac{1}{2}} dt + it \left(1 - \frac{C}{t^4}\right)^{\frac{1}{2}} \sigma_1 \right) \wedge (t\sigma_2 + it\sigma_3)$$

3 Boundary Conditions

Let us consider the possible values of our two real-valued parameters (T, C). Out of the nine combinations of possible cases $T < 0, T = 0, T > 0$, and $C < 0, C = 0, C > 0$, clearly equation (14) only has real solutions, and this metric is only defined if $Tr^2 + C > 0$, so straight away, we may throw out two cases $(T < 0, C \leq 0)$. Also we may discard cases where $T = 0$, since by (13), this implies that the Ricci curvature vanishes, so the universal cover $\tilde{M}$ of M is Calabi-Yau, in particular the canonical bundle is trivial with flat connection i.e. we can choose s such that ψ vanishes, hence this metric just reproduces the product metric.

We summarise the results of this section with the following table:

	$T < 0$	$T = 0$	$T > 0$
$C < 0$	Not defined	Product	Incomplete
$C = 0$	Not defined	Product	Singular
$C > 0$	Incomplete	Product	Complete

Now we show both the cases $(T < 0, C > 0)$ and $(T > 0, C < 0)$ do not define a complete metric for essentially a single reason: that radial geodesics reach $Tr^2 + C = 0$ in finite time, despite this being a degenerate point for the metric in these coordinates.

Proposition 3.1. *The metric defined in* (15) *is complete and not isometric to a product metric iff g_M is complete, and $T > 0, C > 0$.*

In the interest of rigour however, we will prove the following basic lemma for general Riemannian metrics:

Lemma 3.2. *Let r parametrize the interval $I \subset \mathbb{R}$, and $\left(I \times P, g = f(r)^2 dr^2 + g_r\right)$ be a Riemannianian manifold, for some smooth function f, manifold P with smooth 1-parameter family of Riemannian metrics g_r, then the arc-length parametrization of the curves $\gamma_p(t) = (t, p)$ are geodesics, for any $p \in P$.*

Proof. We may re-parametrise g by $\tilde{r} = \int_a^r f(s)ds$, then $d\tilde{r} = fdr$. f must be non-vanishing so that g is non-singular, so $\int_a^r f(s)ds$ is strictly increasing, hence $\tilde{r}(r)$ will be invertible on the entire interval, and the metric $g_{r(\tilde{r})}$ on P will be be smooth. Therefore, it suffices to consider $g = dr^2 + g_r$. Let $\gamma_p' = \partial_r$. We want to show then, that $g\left(\nabla_{\partial_r}\partial_r, X\right) = 0$ for all $X \in \Gamma\left(T(I \times P)\right)$. Now restrict to some open set $U \subseteq P$, so that we may write $X|_{I \times U}$ in an orthonormal basis with respect to $g|_{I \times U}$ i.e. $X|_{I \times U} = a_0 \partial_r + \sum_{i>0} a_i X_i$, for some $a_i \in C^\infty(I \times U)$, $X_i \in \Gamma\left(TU\right)$, such that $g_r\left(X_i, X_i\right) = a_i^{-2}$.

$$g\left(\nabla_{\partial_r}\partial_r, \partial_r\right) = \frac{1}{2}\mathcal{L}_{\partial_r} g\left(\partial_r, \partial_r\right) = 0$$

And by Koszul formula:

$$2g\left(\nabla_{\partial_r}\partial_r, X_i\right) = 2g\left(\left[\partial_r, X_i\right], \partial_r\right) = 0$$

□

Thus if the unique radial geodesic $\gamma_p(t) = (t, p)$ through (r, p) cannot be extended for all t, then the metric will be incomplete. This is independent of choice of coordinates for g on $I \times P$, geodesics are independent of choice of parametrisation. We return to the discussion of the Calabi metric (15), apply this lemma to $L \backslash M \cong P \times \mathbb{R}_{>0}$. Then in order for the metric to be complete, it is necessary that the radial geodesics should be defined for all values of the parametrisation by arc-length. However, in the case $-\frac{C}{T} > 0$, then $r^2 \to -\frac{C}{T}$

will be singular, so the arc-length should tend to infinity. Since length of a curve is parametrization invariant, then the length of these geodesics starting at $(p,b) \in P \times \mathbb{R}_{>0}$ will be:

$$l_\gamma = \left(\frac{1}{m+1}\right)^{\frac{1}{2}} \int_{\left(-\frac{C}{T}\right)^{\frac{1}{2}}}^{b} \left(Tr^2 + C\right)^{-\frac{m}{2(m+1)}} dr$$

This integral in either case is finite: we could see this for example, by noting that $Tr^2 + C$ for $r \geq \left(-\frac{C}{T}\right)^{\frac{1}{2}}$ is bounded below by the line passing through $C = 0$ and $\left(-\frac{C}{T}\right)^{\frac{1}{2}}$, so one may make a linear change of variable to get a finite integral. E.g. in the case $(T > 0, C < 0)$:

$$\begin{aligned} l_\gamma &\leq \left(\frac{1}{m+1}\right)^{\frac{1}{2}} \int_{\left(-\frac{C}{T}\right)^{\frac{1}{2}}}^{b} \left(-C\left(-\frac{C}{T}\right)^{-\frac{1}{2}} r + C\right)^{-\frac{m}{2(m+1)}} dr \\ &= \text{const} \times \int_0^{b'} r^{-\frac{m}{2(m+1)}} dr \\ &= \text{const} \times r^{\frac{m+2}{2(m+1)}}\Big|_0^{b'} < \infty \end{aligned}$$

The case $(T < 0, C > 0)$ is similar. Hence radial geodesics for $(T < 0, C > 0)$ and $(T > 0, C < 0)$ are not defined for all time, so these metrics will not be complete. Moving on to the case $(T > 0, C = 0)$ we have:

$$g = T^{\frac{1}{m+1}} \left(r^{\frac{2}{m+1}} g_M + \frac{1}{m+1}\frac{1}{T} r^{-\frac{2m}{m+1}} \left(dr^2 + r^2\sigma^2\right)\right)$$

Make the change of variable:

$$\tilde{r} = (m+1)^{\frac{1}{2}}\, r^{\frac{1}{m+1}}$$

So we get the metric:

$$g = T^{\frac{1}{m+1}} \left(\frac{1}{T} d\tilde{r}^2 + \frac{1}{T}\frac{\tilde{r}^2}{(m+1)^2}\sigma^2 + \frac{\tilde{r}^2}{(m+1)} g_M\right)$$

I.e. with this metric has the form of the Riemannian cone over (P, g_P), $\left(C(P), dr^2 + r^2 g_P\right)$, with a metric on P given by:

$$g_P = \frac{1}{(m+1)^2}\sigma^2 + \frac{1}{(m+1)} T g_M$$

Recall that Riemannian cones are always singular at the origin unless $(P, g_P) = \left(S^{2m+1}, g_{S^{2m+1}}\right)$, the unit sphere in $\mathbb{R}^{2m+2}$ with the standard metric. On the other hand, the situation is somewhat better than the previous case, as here our manifold only has potential singularities at the isolated point $\tilde{r} = 0$. In general, Riemannian manifolds with Ricci-flat metrics and finitely many isolated conical singularities are known as *conifolds* [11]. However we still may be able to de-singularise, at least if we choose M, and our parameter T correctly. We will return to what choice of parameters amounts to later, but for now, note that P is an S^1 bundle over M, so the best we could hope to have is that M is an S^1 quotient of S^{2m+1}. If we choose the Fubini-Study metric on $\mathbb{CP}^m \cong S^{2m+1}/S^1$ with normalisation such that the scalar curvature $T = \frac{1}{m+1}$, then locally, we retrieve the standard flat metric on $\mathbb{C}^{m+1} \cong \mathbb{R}^{2m+2}$, up to overall scaling. However, recall the the canonical bundle of $\mathbb{CP}^m$ is isomorphic to $\mathcal{O}(-m-1)$, the $(m+1)^{\text{th}}$ power of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{CP}^m$, so in this case $P \cong S^{2m+1}/\mathbb{Z}_{m+1}$, with $\mathbb{Z}_{m+1}$ acting along the Hopf fibre. Hence, the metric we get is globally defined on the orbifold $\mathbb{C}^{m+1}/\mathbb{Z}_{m+1}$.

Finally, we come to the case $(T > 0, C > 0)$. Now S^1 acts isometrically on $P \times \{r\}$ for fixed r, so to check completeness, it suffices to check completeness for radial geodesics. Again, this amounts to the divergence of the integral:

$$l_\gamma = \left(\frac{1}{m+1}\right)^{\frac{1}{2}} \int_a^b \left(Tr^2 + C\right)^{-\frac{m}{2(m+1)}} dr$$

in the limits $b \to \infty$ and $a \to 0$.

Proposition 3.3. *The metric* (15) *with* $T > 0, C > 0$ *is not flat.*

Proof. There are many ways of showing this, but we give a quick proof: recall that [2, p.144], that the fix point-set $\mathrm{Fix}(S) \subset M$ for a set of isometries $S \subset \mathrm{Iso}\,(M, g)$ is a totally-geodesic sub-manifold, i.e. the second fundamental form $\mathrm{II}\,(X, Y)$ vanishes on $\mathrm{Fix}(S)$. For arbitrary sub-manifolds $N \subset M$, this is given by:

$$\mathrm{II}\,(X, Y) := \nabla^N_X Y - \nabla_X Y$$

where ∇^N is the connection on N induced by the Levi-Civita connection ∇ on M. If R^N is the curvature tensor of ∇^N, then:

$$g\left(R\left(X, Y\right) W, Z\right) = g\left(R^N\left(X, Y\right) W, Z\right) - g\left(\mathrm{II}\left(Y, Z\right), \mathrm{II}\left(X, W\right)\right) + g\left(\mathrm{II}\left(X, Z\right), \mathrm{II}\left(Y, W\right)\right)$$

Thus the Riemann curvature tensor on a totally-geodesic sub-manifold agrees with the Riemann curvature tensor of the manifold itself, for $X, Y, W, Z \in \Gamma(TN)$. In particular, in our example, if we let $S = U(1)$, with the action induced from the bundle structure: i.e. sending $(x, z) \mapsto \left(x, ze^{i\theta}\right)$, then $M = \mathrm{Fix}(S)$, and $g|_M = C^{\frac{1}{m+1}} g_M$. By our previous analysis, if g_M is flat, then so is g, thus (L, g) is flat if and only if (M, g_M) is. ☐

3.1 Interpreting parameters, and asymptotics.

The construction of the Calabi-Yau metric (15), appeared in an apparent two-parameter family, with constants T, C. However, we will show that these parameters only alter the metric up to an overall re-scaling.

First, recall that we defined T as $\frac{\mathrm{scal}(g_M)}{2m}$. Under a rescaling of the base $g_M \longmapsto \lambda^2 g_M$ by a constant λ, we get $T \longmapsto \lambda^{-2} T$. Making the re-parametrization $r \mapsto \frac{r}{T^{\frac{1}{2}}}$ in (15), we get the metric:

$$g = \left(r^2 + C\right)^{\frac{1}{m+1}} g_M + \frac{1}{T}\frac{1}{m+1}\left(r^2 + C\right)^{-\frac{m}{m+1}}\left(dr^2 + r^2\sigma^2\right) \tag{17}$$

So the only effect of rescaling g_M, and thus T, is changing the overall scale of this metric.

Similarly, notice that equation (14) is homogenous, thus we can expect solutions to be homogeneous, and indeed, if we re-parametrize:

$$r \mapsto rC^{\frac{1}{2}}$$

Then we get:

$$g = C^{\frac{1}{m+1}}\left(\left(Tr^2 + 1\right)^{\frac{1}{m+1}} g_M + \frac{1}{m+1}\left(Tr^2 + 1\right)^{-\frac{m}{m+1}}\left(dr^2 + r^2\sigma^2\right)\right) \tag{18}$$

Thus C represents an overall scale of this metrics, and the manifold M sits inside L, restricted to the zero-section $r = 0$, with size $C^{\frac{1}{m+1}}$. This also makes explicit what the asymptotics of this metric are: the limit $r \to \infty$ is the limit $C \to 0$, which is the calculation done in the previous section.

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