ALE, ALF, and Abelian Instantons

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1 Bianchi-IX and Isometry

Given some action of a Lie group G on a Riemannian manifold (M, g), $M \ni p \longmapsto hp$, $h \in G$, define (M, g) be (left-) G-invariant if that the induced action on TM acts isometrically. In other words: for fixed $h \in G$ write the map above $L_h : M \longrightarrow M$. If g_p is the Riemannian metric on T_pM, then a G-invariant metric satisfies $(L_h)_* g_m = g_{hp}$. The **Bianchi-IX** metrics are a class of $SU(2)$-invariant metrics on 4-manifolds, such that the manifold, on some dense open subset, is diffeomorphic to $I \times S^3$, with $I \subset \mathbb{R}$. They take the following form:

$$g = f_0^2(r)dr^2 + f_1^2(r){\sigma_1}^2 + f_2^2(r){\sigma_2}^2 + f_3^2(r){\sigma_3}^2 \tag{1}$$

Here we have the local basis of $SU(2)$ left-invariant 1-forms $\sigma_1, \sigma_2, \sigma_3$, and functions f_i to be determined. From a physics point of view, one reason these metrics are interesting because this $SU(2)$-invariance is locally equivalent to rotational (i.e. $SO(3)$)-invariance, so provide possible rotationally symmetric solutions to Einstein's equations for gravity in Euclidean space. There are many interesting metrics of the Bianchi-IX type, but we will focus on a particular subset of these.

Now, if we assume our manifold is orientable, then there is a generic point-wise isometric action by $SO(4)$ in the orthonormal frame-bundle given by the adjoint representation. However, we may want to impose isometries in some open set: recall that $SO(4) \cong (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$, where we can identify each $SU(2)$ as acting on $\mathbb{R}^4$ on the left/right respectively. From the form of the metric, we have already assumed an $SU(2)_L$-isometry, but we may impose an additional $U(1) \hookrightarrow SU(2)_R$ right-invariance, to obtain a metric of the following form [1]:

$$g = dr^2 + \varphi^2(r)\left(\psi^2(r){\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2\right) \tag{2}$$

We have also re-parametrized our metric such that the r-coordinate axis is a radial geodesic. In what follows, we will only consider metrics with these isometries, and we will examine their behaviour in terms of functions ψ, φ. Later on, we will re-parametrise this metric where necessary, but we will be somewhat explicit with our calculations, and hopefully this transparency will clarify some of the geometric intuition.

In doing so, we will see the prototypical examples of two types of geometries: ALE and ALF, intuitively: these are metrics with asymptotic behaviour as scalar multiples of the canonical flat metrics, on $\mathbb{R}^4$, and $\mathbb{R}^3 \times S^1$ respectively[1]. Again, there is much interest in these type of spaces from physics, from which the two examples presented are drawn[2]. As much of the work on these types of spaces is relatively recent, there are differing definitions in literature, but one may consult L.Foscolo for a more precise statement. We will omit them as we will not use them, but rather we will highlight the key features in the construction of these metrics.

Finally, using our previous calculations, we will construct *abelian instantons* over these two spaces: these are $U(1)$ principal bundles with anti-self dual curvature 2-forms. Again this question has both physical and geometric motivation: one reason is that these two forms minimise the *Yang-Mills functional* S_{YM}. For an $SU(n)$ principal-bundle, over 4- manifold X, which has curvature form F, the Yang-Mills functional is defined as:

$$S_{YM} = \int_X \mathrm{Tr}\left(F \wedge \star F\right)$$

In the abelian case, these instantons correspond to a solution to the classical equations of motion for electromagnetism over the manifold.

[1] These are actually the first two in a family of spaces: ALE, ALF, ALG, ALH, with the flat metrics on $\mathbb{R}^{4-k} \times (S^1)^k$ for $0 \leq k \leq 3$.

2 Eguchi-Hanson and ALE geometry

Now we have a metric (2), we choose the orthonormal co-frame $\{e^0, e^1, e^2, e^3\} = \{dr, \psi\varphi\sigma_1, \varphi\sigma_2, \varphi\sigma_3\}$. Using these, we then define the triple, $i \in \{1, 2, 3\}$, of 2-forms:

$$\omega^i = e^0 \wedge e^i + \sum_{j,k} \frac{1}{2}\varepsilon_{ijk} e^j \wedge e^k \tag{3}$$

In the orthonormal frame, these define endomorphisms of the tangent space by $\omega^i(\cdot\,,\cdot) = g(J_i\,\cdot, \cdot)$:

$$J_1 = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \quad J_2 = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix} \quad J_3 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

This is an almost hypercomplex structure: point-wise[2], $\{J_1, J_2, J_3\}$ obey the quaternion relations $J_i^2 = -1$ and $J_1 J_2 = J_3 = -J_2 J_1$. More generally, for the anti-commutator bracket $\{\cdot, \cdot\}$, we have $\{J_i, J_j\} = 0$ for $i \neq j$.

It is also worth mentioning, there is an S^2 of almost-complex structures given by $J_a = a_1 J_1 + a_2 J_2 + a_3 J_3$ with $(a_1, a_2, a_3) \in S^2$. The claim is easily verified:

$$\begin{aligned} J_a^2 &= \sum_i \sum_j a_i J_i a_j J_j \\ &= \sum_{i=j} a_i J_i a_j J_j + \sum_{i<j} \{a_i J_i, a_j J_j\} \\ &= \sum a_i^2 J_i^2 = -1 \end{aligned}$$

In our calculations, we choose the normalisation of σ_i, such that their exterior differential algebra obeys $d\sigma_i = -\varepsilon_{ijk}\sigma_j \wedge \sigma_k$, e.g. $d\sigma_1 = -2\sigma_2 \wedge \sigma_3$. Let us focus on ω^1 for a moment. From the following:

$$d\omega^1 = 2\varphi\,(\varphi' + \psi)\, dr \wedge \sigma_2 \wedge \sigma_3 \tag{4}$$

We see that $\left(g, J_1, \omega^1\right)$ is an almost Kähler structure iff:

$$d\omega^1 = 0 \Leftrightarrow \varphi' = -\psi \tag{5}$$

Of course, we exclude the trivial solution $\varphi \equiv 0$, since it is degenerate for this metric. Also, we have:

$$d\omega^2 = \varphi\,(2 + 2\varphi'\psi + \psi'\varphi)\, dr \wedge \sigma_3 \wedge \sigma_1 \tag{6}$$

And similarly for $d\omega^3$. So we have further condition if $\left(g, J_i, \omega^i\right)$ is to define an hyperKähler structure:

$$d\omega^1 = d\omega^2 = d\omega^3 = 0 \Leftrightarrow \begin{cases} \varphi' + \psi & = 0 \\ 2 - 2\varphi'^2 - \varphi''\varphi & = 0 \end{cases} \tag{7}$$

This set of ODEs has explicit solutions- one may make a substitution to find:

$$d\omega^1 = d\omega^2 = d\omega^3 = 0 \Leftrightarrow \begin{cases} \psi = -\varphi' \\ \varphi'^2 = 1 - k\varphi^{-4} \end{cases} \tag{8}$$

This metric is known as the **Eguchi-Hanson** metric, and we can write out explicit solutions for it, by changing parametrization:

$$\begin{aligned} t &= \varphi(r) \Rightarrow \\ dt &= \frac{d\varphi}{dr} dr = -\psi dr \end{aligned}$$

So rewriting the metric (2), we get:

$$\begin{aligned} g_{EH} &= \frac{1}{\psi^2} dt^2 + t^2 \left(\psi^2 \sigma_1^2 + {\sigma_2}^2 + {\sigma_3}^2\right) \\ &= \left(1 - \frac{k}{t^4}\right)^{-1} dt^2 + t^2 \left(\left(1 - \frac{k}{t^4}\right)\sigma_1^2 + {\sigma_2}^2 + {\sigma_3}^2\right) \end{aligned}$$

[2] The condition of orientability on X is important if these structures are to be globally defined.

From this parametrisation, we see that its asymptotic behaviour as $t \to \infty$ is that of the Euclidean metric on $\mathbb{R}^4$. Hence this metric is of type **ALE**- asymptotically locally Euclidean. This description leaves the possibility for globally non-Euclidean behaviour of the manifold: in fact, with suitable boundary conditions the Eguchi-Hanson defines a smooth metric on TS^2 for $k > 0$, and with $k = 0$ there is the standard Euclidian metric on the orbifold $\mathbb{R}^4/\mathbb{Z}_2$. These types of spaces were classified by Kronheimer in [3] as quotients $\mathbb{R}^4/\Gamma$, where $\mathbb{R}^4$ has the quaternion structure, and Γ is some finite subgroup of $SU(2)$: in this example, clearly we have $\Gamma \cong \mathbb{Z}_2$. One might wonder how this relates to the metric on TS^2: this is because the classification also requires a resolution of singular points on R^4/Γ, but we will not discuss this here. One final point to note is that a reparametrisation $t \mapsto k^{\frac{1}{4}}t$ gives the metric up to scale $k^{\frac{1}{2}}$:

$$g_{EH} = \left(1 - \frac{1}{t^4}\right)^{-1} dt^2 + t^2 \left(\left(1 - \frac{1}{t^4}\right)\sigma_1^2 + {\sigma_2}^2 + {\sigma_3}^2\right)$$

The point is that we may always pick $k = 1$ as this just corresponds to rescaling the metric: so there is really only one metric we are describing here.

3 Taub-NUT and ALF geometry

3.1 Anti-Self Daul metrics

Recall now that for any 4-manifold (M, g), there is a splitting $\bigwedge^2 (T^*M) = \bigwedge^+ (T^*M) \oplus \bigwedge^- (T^*M)$, induced by the Hodge star operator $\star$. These correspond to the $+1, -1$ eigenspaces at each point of the operator, since $\star^2 = 1$, and we can identify these spaces with the splitting of the Lie algebra $\mathfrak{so}(4) = \mathfrak{so}(3) \bigoplus \mathfrak{so}(3) = \mathfrak{su}(2) \bigoplus \mathfrak{su}(2)$. Notice that the ω_i just defined are a basis for $\bigwedge^+ (T^*M)$ in this splitting: let $dV := e^0 \wedge e^1 \wedge e^2 \wedge e^3$, the standard oriented volume form on M induced by g. Then we have:

$$\omega^i \wedge \omega^j = 2\delta^{ij} dV$$

A key property of the Eguchi-Hanson metric of the last example is that it is **anti-self dual**[3] in the sense that $\star F^{ij} = -F^{ij}$, where the curvature F^{ij} of the Levi-Civita connection on M is a section of the bundle:

$$F \in C^\infty \left(M, \, \mathrm{End}(T^*M) \bigotimes \bigwedge^2 (T^*M) \right)$$

Anti-self duality corresponds then to the condition that the induced Levi-Civita connection, when restricted to the bundle $\bigwedge^+ (T^*M)$, is flat. For Eguchi-Hanson, it is no coincidence that there is also a hyperKähler[4] structure (3) - recall also that $\pi_1(TS^2) = 0 = \pi_1(\mathbb{R}^4 \setminus 0)$, so the following lemma applies (stated in [3]):

Lemma 1. *Let (M, g) be a Riemannian 4-manifold such that $\pi_0(M) = \pi_1(M) = 0$. If the curvature F of the Levi-Civita connection ∇ of g satisfies $\star F^{ij} = -F^{ij}$, then (M, g) is hyperKähler.*

Proof. There is a bijection (see [4]) between the set flat connections (P, A) on any principal-bundle $G \hookrightarrow P \to M$, and the set $\mathrm{Hom}(\pi_1(M, x), G)$, for some base-point $x \in M$. This assigns $(P, A) \mapsto \rho_A$, where:

$$\rho_A : \gamma_p \mapsto h_{A,\gamma}(p) \in Hol_p(P, A) \subseteq G$$

Here γ_p is the unique horizontal lift of $\gamma \in \mathcal{L}_x(M)$, the space of based loops at x, and ρ_A descends to a well-defined map on $[\gamma] \in \pi_1(M, x)$. So if M is simply-connected, and G connected, then we may take ρ_A to be the map:

$$\rho_A : \gamma_p \mapsto h_0 \in Hol_p(P, A) \quad \forall p \in P \mid_x$$

Without loss of generality, we may take $h_0 = Id_G$, since otherwise we may apply the automorphism $L_{h_0^{-1}} : P \longrightarrow P$ that sends $\gamma_p \mapsto \gamma_p(0)h_0^{-1}$- hence for any based loop $\gamma \in \mathcal{L}_x(M)$, we have that its holonomy is trivial. Identify $P \mid_x \cong G$ then one may obtain a trivialisation of P, by constructing a globally defined isomorphism $P \mid_x \cong P \mid_y \; \forall y \in M$ as follows: take $\eta : [0, 1] \longrightarrow M$, be some path from x to y, with unique horizontal lift η_g, for $g \in P \mid_x$. Then for fixed define the parallel transport map:

$$T_\eta : P \mid_x \longrightarrow P \mid_y$$
$$T_\eta : g \mapsto \eta_g(1)$$

Since the horizontal lift is unique this is well-defined for given g and η. It also clearly everywhere invertible by $T_{\eta^{-1}}$, so defines an isomorphism. However this map is independent of path η, since if $T_{\eta'}$ is some other path, then $T_\eta T_{\eta'}^{-1} = T_{\eta\eta'^{-1}} = Id_G$ as $\eta\eta'^{-1} \in \mathcal{L}_x(M)$. Thus we may trivialise $(P, A) \cong (M \times G, A_{\mathrm{triv}})$, where A_{triv} is the trivial connection.

[3]There is some ambiguity about this definition in the literature: some might say this property is equivalent to being anti-self dual *and* Ricci-flat [2].

[4]The point-wise structure as defined turns out to be necessarily integrable.

Now we have a trivilisation of the any principal bundle associated to M, we use that [4] the Levi-Civita connection is the unique connection on the (oriented) orthonormal frame bundle $P_{SO(4)}$ such that the induced derivative for sections of T^*M is torsion free. So if we use the isomorphism $\bigwedge^2(T^*M) \cong P_{SO(4)} \times_{Ad} \mathfrak{so}(4)$, then by some automorphism of the frame bundle we get the splitting $\bigwedge^2(T^*M) \cong P_{SO(4)} \times_{\rho_1 \oplus \rho_2} \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where ρ_1, ρ_2 are 3-dimensional representations of $SO(4)$, and furthermore, we may identify $\bigwedge^+(T^*M) \cong P_{SO(4)} \times_{\rho_1} \mathfrak{so}(3)$. Since automorphisms preserve the space of flat connections, then the connection restricted to this bundle is flat. By previous discussion, we get that $\rho_1 \cong \mathrm{Id}$, and we may trivialise this bundle. Similarly, the space of covariant constant sections of a bundle is preserved by automorphisms. So for some $x \in M$, pick an orthonormal triple of 2-forms $\omega^i \mid_x$ in $\bigwedge^+(T_x^*M)$, then this will define a covariant constant trivialisation of $\bigwedge^+(T_x^*M)$ the whole of M, via parallel transport.

Hence for $\omega := (\omega^1, \omega^2, \omega^3)$, $\nabla\omega = 0$, and a quick calculation verifies for all $u, v \in C^\infty(TM)$:

$$\begin{aligned} \nabla(\omega(u,v)) &= \nabla(g(Ju,v)) \\ &= g((\nabla J)u, v) + g(J\nabla u, v) + g(Ju, \nabla v) \\ &= g((\nabla J)u, v) + \nabla(\omega(u,v)) \\ \therefore \nabla J &= 0 \end{aligned}$$

This parallel triple verifies the claim that we have a hyperKähler structure. □

Remark. *The converse is also true (up to sign)- i.e. every simply-connected manifold with a hyperKähler structure is either anti-self or self dual depending on choice of orientation.*

To say something about how to show the converse: if $(\omega^1, \omega^2, \omega^3)$ are the two-forms associated to a hyperKähler structure (J_1, J_2, J_3), then they are covariant constant. As such they provide covariant constant non-vanishing sections of the bundle $\bigwedge^2(T^*M)$. If we pick a local orthonormal trivialisation of T^*M with basis e^i, then we get connection-forms α^{ij}, where $\nabla(e^i) = e^j \otimes \alpha^{ij}$. The induced Levi-Civita connection on $e^i \wedge e^j \in \bigwedge^2(T^*M)$ can be written:

$$\nabla(e^i \wedge e^j) = e^k \wedge e^j \otimes \alpha^{ki} + e^i \wedge e^k \otimes \alpha^{kj}$$

Then the splitting of $\bigwedge^2(T^*M)$ induces a splitting $\alpha^{ij} = \alpha_+^{ij} + \alpha_-^{ij}$. It remains to be shown that the S^2 of forms containing ω_+ are the only possible hyperkahler structures, and thus form a basis for $\bigwedge^+(T * M)$. Since the structures are parallel by definition, this would imply that $\alpha_+^{ij} = 0$ and thus the curvature vanishes on $\bigwedge^+(T * M)$.

Since we have found one anti-self dual metric, let us see if we can find another: let us return to the action of $(SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ on M. If there is an isometric action on g for T^*M, we should also expect there to be an isometric action on the induced metric on $\bigwedge^2(T^*M)$. If furthermore we know that there is a hyperKähler structure, then it should preserve the 2-sphere of Kähler forms J_a: i.e. there is a homomorphism $\rho : SU(2) \longrightarrow SO(3)$. If we fix an isomorphism $T_x^*M \cong \mathbb{R}^4 \cong \mathbb{R} \oplus \mathfrak{su}(2)$ then the induced parallel transport map should fix $\bigwedge^+(T^*M)$, so by suitable coverings, we have a 3-dimensional representation ρ of $S^3 = SU(2)$.

3.2 Taub-NUT

Now recall some representation theory of $SU(2)$: the irreducible representations are isomorphic to the standard representation on $\mathrm{Sym}^k(\mathbb{C}^2)$, so the three-dimensional representations of $SU(2)$ are isomorphic to either the trivial representation or Ad, the adjoint representation. In the previous example of Eguchi-Hanson, we saw the trivial representation, i.e. the hyperKähler structure $\tilde{\omega}$ satisfies $\tilde{\omega} = I\omega$. However, by the above reasoning, we should also look for be a metric satisfying an $SU(2)/SO(3)$-invariance coming from the adjoint representation. A useful way to write the adjoint representation on $\mathfrak{su}(2) \cong \mathbb{R}^3$ is using the quaternions $\mathbb{H} := \{a_0 + a_1Q_1 + a_2Q_2 + a_3Q_3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$. We have $q \in S^3 \subset \mathbb{H}$ acting on $\omega \in \mathfrak{su}(2) \cong \mathrm{Im}\mathbb{H}$ via. conjugation:

$$\tilde{\omega} = q\omega\bar{q}$$

To be a hyperKähler structure then, we require $d\tilde{\omega} = 0$. Expanding we get:

$$\begin{aligned}
d\tilde{\omega} &= d(q\omega\bar{q}) \\
&= dq \wedge \omega\bar{q} + qd\omega\bar{q} + q\omega \wedge d\bar{q} \\
&= dq \wedge \omega\bar{q} + qd\omega\bar{q} - q\omega \wedge \bar{q}dq\bar{q} = 0 \implies \\
\bar{q}d\tilde{\omega}q &= \bar{q}dq \wedge \omega + d\omega - \omega \wedge \bar{q}dq \\
&= d\omega + [\bar{q}dq, \omega] \\
&= \nabla_q(\omega) = 0
\end{aligned}$$

Here ∇_q is the connection defined by the metric, since $\bar{q}dq$ is a left-invariant 1-form, the Maurier-Cartan form of q at the identity. Since we can fix a point $x \in M$, and an isomorphism $T_xM \cong \mathbb{R}^4 \cong \mathbb{R} \oplus \mathfrak{su}(2)$, we set this to take values:

$$\bar{q}dq = \sigma_1 Q_1 + \sigma_2 Q_2 + \sigma_3 Q_3$$

We continue

$$\begin{aligned}
0 = d\omega + [\bar{q}dq, \omega] &= d\omega + \left[\sum_i \sigma_i Q_i, \sum_j \omega_j Q_j\right] \\
&= \sum_k d\omega_k Q_k + \sum_{i,j} \sigma_i \wedge \omega_j \, [Q_i, Q_j] \\
&= \sum_k \left(d\omega_k + 2\sum_{i,j} \varepsilon_{ijk}\sigma_i \wedge \omega_j\right) Q_k \implies \\
0 &= d\omega_k + 2\sum_{i,j} \varepsilon_{ijk}\sigma_i \wedge e^0 \wedge e^j + \sum_{i,j,l,m} \varepsilon_{ijk}\varepsilon_{jlm}\sigma_i \wedge e^l \wedge e^m \\
&= d\omega_k + \sum_{i,j} 2\varepsilon_{ijk}\sigma_i \wedge e^0 \wedge e^j \\
&= d\omega_k - \sum_{i,j} 2\varepsilon_{kij} dr \wedge \sigma_i \wedge e^j
\end{aligned}$$

Then, using the formulae for $d\omega_k$ given in the previous section, we get:

$$\begin{aligned}
\nabla_1\omega &= 2\varphi\left(\varphi' + \psi - 2\right) dr \wedge \sigma_1 \wedge \sigma_2 &&= 0 \\
\nabla_2\omega &= \varphi\left(2\varphi'\psi + \varphi\psi' - 2\psi\right) dr \wedge \sigma_2 \wedge \sigma_3 &&= 0 \\
\nabla_3\omega &= \varphi\left(2\varphi'\psi + \varphi\psi' - 2\psi\right) dr \wedge \sigma_3 \wedge \sigma_1 &&= 0
\end{aligned}$$

So we get the ODE system:

$$d\tilde{\omega} = 0 \Leftrightarrow \begin{cases} \varphi\left(\varphi' + \psi - 2\right) & = 0 \\ \varphi\left(2\varphi'\psi + \varphi\psi' - 2\psi\right) & = 0 \end{cases} \tag{9}$$

Let us re-parametrize to obtain explicit solutions, introduce variable t:

$$\begin{aligned}
r(t) &:= \int_0^t \varphi(s)ds \implies \\
\frac{dr}{dt} &= \varphi(r(t)) \implies \\
\frac{d}{dr} &= \frac{1}{\varphi}\frac{d}{dt}
\end{aligned}$$

Then we get:

$$d\tilde{\omega} = 0 \Leftrightarrow \begin{cases} \frac{1}{2\varphi}\frac{d}{dt}\left(\varphi^2\right) + \varphi\psi - 2\varphi & = 0 \\ \frac{1}{\varphi}\frac{d}{dt}\left(\varphi^2\psi\right) - 2\varphi\psi & = 0 \end{cases} \tag{10}$$

The second equation gives us $\varphi^2\psi = Ae^{2t}$ for some constant A. Now writing $\frac{d\psi}{dt} := \dot{\psi}$, we get:

$$d\tilde{\omega} = 0 \implies \begin{cases} \dot{\varphi} - (2-\psi)\varphi & = 0 \\ \dot{\psi} - \left(2\psi - 2\frac{\dot{\varphi}}{\varphi}\psi\right) & = 0 \end{cases} \implies \begin{cases} \dot{\varphi} - (2-\psi)\varphi & = 0 \\ \dot{\psi} - 2\psi(\psi - 1) & = 0 \end{cases} \tag{11}$$

Now we have de-coupled the second equation for ψ, it becomes readily integrable:

$$\psi^{-1}(\psi-1)^{-1}\dot{\psi}=2 \Longrightarrow$$
$$\psi=\left(1+Be^{2t}\right)^{-1}$$

For some constant of integration $-B$. Using we have two first order equations:

$$\begin{cases}\varphi^2\psi = Ae^{2t} \\ \psi=\left(1+Be^{2t}\right)^{-1}\end{cases} \Longrightarrow \begin{cases}\varphi^2 = Ae^{2t}\left(1+Be^{2t}\right) \\ \psi=\left(1+Be^{2t}\right)^{-1}\end{cases} \tag{12}$$

Make one final change of variable u:

$$u=Be^{2t} \Longrightarrow$$
$$du = 2udt \Longrightarrow$$
$$dr = \frac{\varphi}{2u}du$$

$$\begin{cases}\varphi^2 = \frac{A}{B}u\left(1+u\right) \\ \psi=\left(1+u\right)^{-1}\end{cases} \tag{13}$$

And then the metric g becomes:

$$g=\frac{A}{B}\frac{1+u}{4u^2}\left(du^2+4u^2\left({\sigma_2}^2+{\sigma_3}^2\right)\right)+\frac{A}{B}\frac{u}{1+u}{\sigma_1}^2$$

Clearly the choice of parameters A, B correspond to an overall scaling of the metric by a constant $\frac{A}{B}$. Setting this to be 4, we get the **Taub-NUT** metric, [2]:

$$g_{TN}=\left(\frac{1}{u}+1\right)\left(du^2+4u^2\left({\sigma_2}^2+{\sigma_3}^2\right)\right)+4\left(\frac{1}{u}+1\right)^{-1}{\sigma_1}^2 \tag{14}$$

This metric, besides being a critical example of a number of geometric phenomena, is an example of an asymptotically locally flat (**ALF**) geometry of $\mathbb{R}^3\times S^1$. Let us verify this claim for the parameter u: in Euclidean space, the induced metric on $S^n\left(R\right)$, the sphere of radius R, is $g_R = R^2 ds_n^2$, where ds_n^2 is the standard metric on $S^n\left(1\right)$. Now the Riemannian submersion of the Hopf fibration gives:

$$\left(SU(2),\ \sigma_1^2+\sigma_2^2+\sigma_3^2\right)\cong\left(S^3(1), ds_3^2\right)\longrightarrow\left(S^2(1),\frac{1}{4}ds_2^2\right)$$

Since $\langle\sigma_1\rangle$ is the dual subspace generated by the S^1 action at $T_I^*SU(2)$ in the Hopf fibration, we have then that $du^2+4u^2\left({\sigma_2}^2+{\sigma_3}^2\right)$ gives the standard metric on $\mathbb{R}^3$. We can now see explicitly that this metric has asymptotic behaviour as $u\to\infty$ of the product metric- of (up to some scalar) the Euclidean and the standard (flat) metric on $\mathbb{R}^3\times S^1$ respectively. In this example, we get the asymptotic fibration of $\mathbb{R}^3$ with an $S^1(2)$.

While we have examined the asymptotic geometry of this metric, there remains a question of what total space this metric can actually be defined on: I claim that Taub-NUT metric defines a complete metric on $\mathbb{R}^4$. Let us consider what this means in terms of our original parametrisation (2). Let $r=\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}$ be the radial coordinate on $\mathbb{R}^4$, then it is automatically continuous on all of the domain, so checking completeness is equivalent to checking continuity of t, which is guaranteed if $\varphi(r)$ is at least differentiable. Furthermore, if we want to make sure that this metric is smooth, we only need to make sure that $\psi(r)^2, \varphi(r)^2$ are smooth near the origin, since r is only non-smooth there. We can assume ψ, φ are already smooth functions of their arguments. Finally, we will need to check that our boundary conditions can actually be satisfied by the ODE system (9). Based on these requirements, let us choose the boundary conditions as follows:

$$\begin{cases}\varphi(0)=0 & \psi(0)=1 \\ \varphi'(0)=1 & \psi'(0)=0 \\ \varphi^{(even)}(0)=0 & \psi^{(odd)}(0)=0\end{cases} \tag{15}$$

It is easy to verify that (9) is satisfied at $r=0$. As to the smoothness of ψ, φ with respect to the coordinates, the parity conditions guarantee that φ^2 and $\varphi^2\psi^2$ are functions of r^2 near the origin. Finally, the derivative conditions ensure that:

$$\left.\frac{d\varphi}{dr}\right|_{r=0}=\left.\frac{d(\varphi\psi)}{dr}\right|_{r=0}=1$$

Thus we get a vertical tangent condition: by suitable Taylor approximation (we will further assume here that ψ, φ are real analytic) near $r = 0$, we can obtain the metric:

$$g = dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + O\left(r^4\right)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

We get the Euclidean metric plus some higher order terms in $\sigma_1, \sigma_2, \sigma_3$ near the origin- thus we avoid conical singularities. Furthermore, I claim also that (15) are necessary conditions. From our parametrisation above, it is clear that the only potentially singular points will be at $\lim_{u \to 0}$: i.e. from (8), $\psi > 0$ for all u, and $\varphi \to 0$ only as $u \to 0$. To get this back to boundary conditions in terms of r, consider (9). Since these equations are homogeneous ODE system, i.e. with no explicit r dependence, the solution set is invariant under translations (avoiding singularities). Once we have obtained the explicit solutions in terms of u, as we have $\varphi \to 0$ we have $r \to r_0$, we may shift $r \mapsto r - r_0$ so that $\varphi \to 0$ as $r \to 0$. From (8) then we must have also $\psi \to 1$. Once we have this, then the only potential singular point is at $r = 0$, i.e. we have a Riemannian cone. On the topological level, we have the standard (non-compact) cone $C(M^3)$ over some 3-dimensional space M^3, and we equip this space with the metric $dr^2 + r^2 g_M$. Topologically, for this to be a smooth manifold, on must have $M^3 \cong S^3$ so that $C(M^3) \cong \mathbb{R}^4$, and geometrically, bounded curvature means we must have g_M has curvature 1. So this metric avoids conical singularities if and only if[5] $\varphi'(0) = 1$. Then the rest of the boundary conditions follow from considering this Taylor expansion.

4 Abelian Instantons

Let us now step back for a moment from Eguchi-Hanson, and Taub-NUT metrics and consider $U(1)$-bundles over these two spaces, i.e. $U(1)$-bundles over $M = TS^2$ or $\mathbb{R}^4$. If we are able to construct such a bundle $U(1) \hookrightarrow P \to M$, we may give P a connection form A, i.e. a map $A : TP \longrightarrow i\mathbb{R}$ satisfying certain equivariance properties. We canonically identify this form as an element $A \in \Omega^1(P)$, then since $U(1)$ is abelian we obtain the curvature 2-form $F = dA$. In order to specify this bundle up to isomorphism, given some space M, it is enough to give the first Chern class, i.e. an element in $c_1(P) \in H^2(M, \mathbb{Z})$, such that $F \in H^2_{dR}(M) \cong H^2(M, \mathbb{R})$ is associated to this Chern class via. the natural pairing:

$$\begin{aligned} \Pi : H_2(M, \mathbb{Z}) \times H^2_{dR}(M) &\longrightarrow \mathbb{R} \\ ([\sigma], [\omega]) &\longmapsto \int_\sigma \omega \end{aligned}$$

The association comes about as follows: an *integral class* $[\omega]$, i.e. a class that satisfies $\Pi([\sigma], [\omega]) \in \mathbb{Z}$ for all $[\sigma] \in H_2(M, \mathbb{Z})$, defines an element $\Pi(\cdot, \omega) \in \mathrm{Hom}(H_2(M, \mathbb{Z}), \mathbb{Z})$. This has an inclusion into $H^2(M, \mathbb{Z}) \cong \mathrm{Hom}(H_2(M, \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(M, \mathbb{Z}), \mathbb{Z})$ via the universal coefficient theorem, thus an integral class $[\omega] \in H^2_{dR}(M)$ defines an element in $H^2(M, \mathbb{Z})$. This element is the first Chern class of this bundle, which characterises the total space of the bundle P up to isomorphisms. Thus, if we are looking to construct such a bundle, it is enough to find a closed integral 2-form on X.

Consider abelian instantons over these spaces: $SU(2)$-invariant 2-forms such that $F = -\star F$. Given our trivialisation of the space of 2-forms, we know at $r \neq 0$, we have:

$$F = \sum_i \alpha_i(r)\omega_-^i$$

So the closed condition implies:

$$dF = \sum_i -\frac{1}{2}\varepsilon_{ijk}\frac{d\alpha_i(r)}{dr} dr \wedge e^j \wedge e^k + \alpha_i(r) d\omega_-^i = 0$$

Let us use the parametrization (2):

$$\begin{aligned} -\alpha_1'(r)\varphi^2 + \alpha_1(r)\left(2\varphi(\psi - \varphi')\right) &= 0 \\ -\alpha_{2,3}'(r)\varphi^2\psi + \alpha_{2,3}(r)\left(\varphi(2 - 2\varphi'\psi - \varphi'\psi)\right) &= 0 \end{aligned}$$

[5]One has to do a little more work here to show that this is the case, by considering curvature.

Then we get solutions depending upon (a-priori) three parameters (A_1, A_2, A_3):

$$\alpha_1(r) = A_1 \exp\left(2\int \frac{\psi - \varphi'}{\varphi} dr\right)$$
$$\alpha_{2,3}(r) = A_{2,3} \exp\left(2\int \frac{2 - 2\varphi'\psi - \psi'\varphi}{\varphi\psi} dr\right)$$

Then, with Eguchi-Hanson, using the conditions (7) and (8):

$$\alpha_1 = A_1 \varphi^{-4}$$
$$\alpha_{2,3} = A_{2,3}\left(\varphi^4 - k\right)^{-2}$$

However boundary conditions for smoothness at the origin $r = 0$ requires that:

$$\alpha_1 = A_1 \varphi^{-4}$$
$$\alpha_{2,3} = 0$$

For integrality then, pick a generator for the homology of TS^2, e.g. the zero section $\sigma \cong S^2$. Since $H_2\left(TS^2, \mathbb{Z}\right) \cong \mathbb{Z}.\sigma$, it is enough to look at the case:

$$C_1 = \int_{S^2} \frac{i}{2\pi} F$$

This is just the pairing defined above, but we have scaled by $\frac{i}{2\pi}$ to agree with convention: i.e. scaled the volume[6] of the generator S^2, and so F is an $i\mathbb{R}$-valued form. The requirement of integrality is such that C^1 is an integer. Recall also we take the convention that the scaling of the Eguchi-Hanson metric is such that $k = 1$, thus we get:

$$F = i\varphi^{-4}\omega_-^1$$

Now let us do the same thing for Taub-NUT: use the parametrization with orthonormal co-frame:

$$\{\left(1+\frac{1}{u}\right)^{\frac{1}{2}} du, 2\left(1+\frac{1}{u}\right)^{-\frac{1}{2}} \sigma_1, 2\left(1+\frac{1}{u}\right)^{\frac{1}{2}} u\sigma_2, 2\left(1+\frac{1}{u}\right)^{\frac{1}{2}} u\sigma_3\}$$

Then in this parametrization:

$$\omega_-^1 = 2du \wedge \sigma_1 - 4\left(u + u^2\right)\sigma_2 \wedge \sigma_3$$
$$\omega_-^2 = (2 + 2u)\, du \wedge \sigma_2 - 4u\sigma_3 \wedge \sigma_1$$
$$\omega_-^3 = (2 + 2u)\, du \wedge \sigma_3 - 4u\sigma_1 \wedge \sigma_2$$

So that we get:

$$d\omega_-^1 = (-8u)\, du \wedge \sigma_2 \wedge \sigma_3$$
$$d\omega_-^2 = (-2 - 2u)\, du \wedge \sigma_3 \wedge \sigma_1$$
$$d\omega_-^3 = (-2 - 2u)\, du \wedge \sigma_1 \wedge \sigma_2$$

Then closed condition gives:

$$-\alpha_1'(u)\left(4u + 4u^2\right) + \alpha_1(u)(-8u) = 0$$
$$-\alpha_{2,3}'(u)(4u) + \alpha_{2,3}(u)(-2 - 2u) = 0$$

Again we get a family of solutions:

$$\alpha_1(u) = A_1 \exp\left(-\int \frac{2}{u+1} du\right)$$
$$\alpha_{2,3}(u) = A_{2,3} \exp\left(-\int \frac{1+u}{2u} du\right)$$

Explicitly:

$$\alpha_1(u) = A_1 (u+1)^{-2}$$
$$\alpha_{2,3}(u) = A_{2,3} u^{-\frac{1}{2}} \exp\left(-\frac{1}{2}u\right)$$

[6] More precisely, so that the canonical metric on the tautological bundle over $\mathbb{CP}^1 \cong S^2$ has unit Chern class.

5 Gibbons-Hawking Ansatz

There is a generalisation of Taub-NUT: we fix an open subset $U \subseteq \mathbb{R}^3$, with a function $h : U \longrightarrow \mathbb{R}$. Then we fix a $U(1)$-principal bundle $\pi : P \longrightarrow U$ over this subset with connection form θ. Then we define the metric:

$$g_{GH} = h\pi_* (g_{\text{Eucl}}) + h^{-1}\theta^2$$

Clearly then h must be a positive function. If we pick coordinates on $U \ni (x_1, x_2, x_3)$ Then, we use the orthonormal co-frame:

$$\{e_0, e_1, e_2, e_3\} = \{h^{-\frac{1}{2}}\theta, h^{\frac{1}{2}}dx_1, h^{\frac{1}{2}}dx_2, h^{\frac{1}{2}}dx_3\}$$

Then the usual hyperKahler triple in this case turns out to be:

$$w_i = dx_i \wedge \theta + h dx_j \wedge dx_k$$

So the requirement that $d\omega_i = 0$, is equivalent to the conditions:

$$\star dh = d\theta$$

Clearly there are some conditions on the function h so that we can find a solution:

$$\star d \star dh = \triangle h = 0$$

I.e. the function is h *harmonic*. In coordinates this is the condition:

$$\sum_i \frac{\partial^2 h}{\partial x_i^2} = 0$$

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