Hamiltonian Actions, Moment Maps, and Convexity

Jakob Stein

23 April 2019

1 Introduction

Symplectic geometry can be seen as generalising Kähler geometry, keeping only the Kähler form ω, and "forgetting" the complex structure. For this more general set-up, one only need a closed two-form, which is non-degenerate i.e. $\iota_X \omega = 0$ if and only if $X = 0$.

A symplectic form can often be defined from the data of a Riemannian metric. In Kähler geometry for example, the complex structure J gives a metric g as follows, for $u, v \in \Gamma(M)$:

$$g(Ju, v) = \omega(u, v)$$

The Riemannian manifolds with holonomy contained in $SU(n)$ or $Sp(n)$ can also be considered as special cases of symplectic manifolds in this way. Many interesting and useful results in these geometries can occur just using the underlying symplectic manifold, such as Gromov's result on J-holomorphic curves [1].

One powerful construction in symplectic geometry, developed by Marsden-Weinstein-Meyer, is called the *symplectic quotient*, and can be used to construct Kähler, and hyperKähler manifolds. The Marsden-Weinstein-Meyer theorem concerns whether a particular quotient preserving the symplectic structure is a smooth manifold. A key ingredient in this theorem is the concept of a *moment map*, associated to the action of a Lie group.

In this note, we will look at some properties of the moment map, particularly when the Lie group is abelian, so called *toric geometry*. This has been applied to the construction of Reimannian manifolds with special holonomy groups, such toric Kähler, Calabi-Yau cones [2, Ch. 5], hypertoric geometry [3], and the construction of G_2-manifolds with a two-torus symmetry [4]. The Calabi-Yau case turns up in physics literature as well, for example in the context of AdS/CFT correspondence and supersymmetry [3], [5]. There is a corresponding notion of an *algebraic moment map* in algebraic geometry, as well as *toric varieties*, see [6], related to notions of stability in geometric invariant theory. Our main reference will be [10].

2 Symplectic Group Actions and Moment Maps

In what follows, will define natural symplectic notions in three categories: groups, vector spaces, and Lie algebras. Assuming one is familiar with the analogous game in Riemannian geometry, we will compare these two games, but as we shall see, they will end up being quite different.

Firstly, let (M, ω) be a symplectic manifold, $\mathrm{Diff}(M)$ the group of diffeomorphisms of M. We define the subgroup of **symplectomorphisms**:

$$\mathrm{Sympl}(M, \omega) := \{\phi \in \mathrm{Diff}(M) \mid \phi^* \omega = \omega\}$$

Let G be an arbitrary group. A **symplectic action** ϕ of G on M is a group homomorphism:

$$\phi : G \to \mathrm{Sympl}(M, \omega) \tag{1}$$

We will write the diffeomorphism associated to $g \in G$ as ϕ_g. The "infinitesimal" picture of symplectic actions by smooth Lie groups leads us to the following definition. If $\Gamma(M)$ is the space of vector fields on M, then a **symplectic vector field** is defined as a linear subspace:

$$\Gamma_{\mathrm{Sympl}}(M, \omega) := \{X \in \Gamma(M) \mid \mathcal{L}_X \omega = 0\} = \{X \in \Gamma(M) \mid d(\iota_X \omega) = 0\} \tag{2}$$

If $\rho_t : M \times I \to M$ is the local flow generated by X, i.e. if we fix $p \in M$, then the curve $\rho_t(p)$ is the unique maximal solution on $I = (a, b) \subseteq \mathbb{R}$ to:

$$\begin{cases} \rho_0(p) = p \\ \frac{d\rho_t(p)}{dt} = X(\rho_t(p)) \end{cases}$$

Then, if for all $p \in M$, we have:

$$(\mathcal{L}_X\omega)_p := \frac{d}{dt}\bigg|_{t=0}((\rho_t)^*\omega_p) = 0 \qquad\qquad \Leftrightarrow \qquad\qquad (\rho_t)^*\omega = \omega$$

So an equivalent condition to $\mathcal{L}_X\omega = 0$ is that the flow ρ_t acts via symplectomorphisms.

We can make the correspondence between symplectic group actions and symplectic vector fields explicit via the lie algebra: given homomorphism $\phi : G \to \mathrm{Sympl}(M, \omega)$ there is an induced homomorphism of lie algebras:

$$\begin{aligned} D\phi : \mathfrak{g} &\to \Gamma_{\mathrm{Sympl}}(M, \omega) \\ X &\mapsto \frac{d}{dt}\bigg|_{t=0} \phi(\exp(-tX)).p \end{aligned} \tag{3}$$

Here, we have the standard lie brackets on these two vector spaces.

Remark 2.1. *Given a lie sub-algebra of complete vector-fields $\mathfrak{g}$, pointwise isomorphic to the lie algebra of some compact, simply-connected Lie group G, we can always locally integrate them to a group action of G. To see this, given some open neighbourhood U of the identity $I_d \in G$, for all $g \in G$ we can write a finite product $g = g_1 \dots g_N$ for $g_i \in U$, so a group action is determined uniquely by its generating vector fields.*

There is a distinguished sub-algebra of $\Gamma_{\mathrm{Sympl}}(M, \omega)$, which uses the symplectic form. Namely, since ω is closed, then $X \in \Gamma_{\mathrm{Sympl}}(M, \omega)$ is equivalent to $\iota_X\omega$ being a closed one-form. The subspace of **Hamiltonian vector fields** is defined such that $\iota_X\omega$ is exact:

$$\Gamma_{\mathrm{Ham}}(M, \omega) := \{X \in \Gamma(M) \mid \exists H \in C^\infty(M) \text{ s.t. } \iota_X\omega = dH\}$$

Lemma 2.2. *Suppose (M, ω) is a connected symplectic manifold, and $C_0^\infty(M)$ is the space of smooth functions H with $\int_M H = 0$. Then there is a isomorphism of vector spaces $C_0^\infty(M) \to \Gamma_{\mathrm{Ham}}(M, \omega)$ that sends $H \mapsto X_H$, where X_H is the unique solution of $dH = \iota_{X_H}\omega$.*

Proof. Clearly, this map is linear, and surjective by definition. If $X_H = 0$, then this implies $dH = 0$, and so $H = 0$ since $H \in C_0^\infty(M)$, hence this map is injective. □

Now clearly, a necessary condition to have an action ϕ corresponding to Hamiltonian vector fields in the image of $D\phi$, is that for every $X \in \mathfrak{g}$, there is an element $\mu^*(X) \in C^\infty(M)$ so that $d\mu^*(X) = \iota_{D\phi(X)}\omega$. Now importantly, with the symplectic action, it is not just a vector space homomorphisms (i.e a linear maps) we want, but a Lie algebra homomorphism.

Lemma 2.3. *Suppose (M, ω) is a symplectic manifold. We have the following Lie algebra inclusions:*

$$\Gamma_{\mathrm{Ham}}(M, \omega) \subseteq \Gamma_{\mathrm{Sympl}}(M, \omega) \subset \Gamma(M)$$

There are two clear differences here to the corresponding notions in Riemannian geometry. The first is that the first inclusion implies, unlike the space of infinitesimal isometries, $\Gamma_{\mathrm{Sympl}}(M, \omega)$ is never finite dimensional by Lemma 2.2. The second is that there is no analogue of $\Gamma_{\mathrm{Ham}}(M, \omega)$: if a one-form θ dual to a vector field via a metric g is exact, then it is Killing if and only it is parallel i.e. $\nabla\theta = 0$, where ∇ is the Levi-Cevita connection. In other words, if $\theta = df$ for some function f, then $\mathrm{Hess}(f) = 0$. However, by taking the trace, this implies f is harmonic, so must be constant if it is bounded.

Lemma 2.4. *Suppose (M, ω) is a compact symplectic manifold and $H^1(M, \mathbb{R}) = 0$. Then $\Gamma_{\mathrm{Ham}}(M, \omega) = \Gamma_{\mathrm{Sympl}}(M, \omega)$.*

Proof. Recall that $X \in \Gamma_{\mathrm{Sympl}}(M, \omega)$ iff $\iota_X\omega$ is closed. Suppose we define a generic Riemannian metric on M, by our co-homological assumption, applying the Hodge theorem says that $\iota_X\omega$ must necessarily by exact, and so $X \in \Gamma_{\mathrm{Ham}}(M, \omega)$. □

Notice all these algebras have natural G action, as does $\mathfrak{g}$ via the adjoint. To see this this action, there is one final Lie algebra structure that still to consider: the algebra of functions on M. This also has a natural G action by the pullback, so as well as the Hamiltonian condition, in order well-defined vector fields on M we will need that equivariance with respect to the G-action. To be a bit more precise about this equivariance, we define the following Lie bracket $\{\cdot, \cdot\}$ for $f, g \in C_0^\infty(M)$, so that the linear map $H \mapsto X_H$ in Lemma 2.2 becomes a Lie algebra *anti-homomorphism*:

$$\{f, g\} := \omega(X_f, X_g) \tag{4}$$

Viewing $\Gamma(M)$ as the space of derivations on $C_0^\infty(M)$, the notion of equivariance and Lie algebra homomorphisms on $C_0^\infty(M)$ coincide. We are now almost ready to define moment maps: let (M, ω) be a symplectic manifold, G connected Lie group with symplectic action ϕ, and Lie algebra $\mathfrak{g}$.

Definition 2.1. A **co-moment map** is a Lie algebra homomorphism $\mu^* : (\mathfrak{g}, [\,\cdot, \cdot]) \to (C^\infty(M), \{\,\cdot, \cdot\})$ making the following diagram commute:

$$\begin{tikzcd} C^\infty(M) \arrow[r, "d"] & \Omega^1(M) \\ \mathfrak{g} \arrow[u, "\mu^*"] \arrow[r, "D\phi"] & \Gamma(M) \arrow[u, "\simeq"'] \end{tikzcd} \tag{5}$$

Here isomorphism between $\Gamma(M) \to \Omega^1(M)$ is given by $X \mapsto \iota_X\omega$.

An equivalent description of this is in terms of its dual, the moment map:

Definition 2.2. A **moment map** is a map $\mu : M \to \mathfrak{g}^*$ such that:

1. $\forall X \in \mathfrak{g}$, $\mu_p(X) = \mu^*(X)|_p$
2. $\mu(\phi_g(p)) = Ad_g^*(\mu(p))$

The quadruple (M, ω, G, μ) is called a **Hamiltonian G-space**.

Again, the second condition of equivariance is precisely the condition that co-moment map be a Lie algebra homomorphism.

We also have the following lemma from [10], which we state without proof:

Lemma 2.5. *Suppose (M, ω) is a symplectic manifold, G compact connected Lie group $H^1_{dR}(G) = H^2_{dR}(G) = 0$ acting by symplectomorphisms. Then there is a unique moment map μ for this action.*

Let us see some important special cases of moment maps:

Example 2.1. Let $G = S^1$, or $G = \mathbb{R}$. Now $\mathfrak{g} \simeq \mathfrak{g}^* \simeq \mathbb{R}$, and the moment map $\mu : M \to \mathbb{R}$, satisfies:

1. On the generator of $\mathfrak{g}$, $X = 1$ and $\mu_p(1) = \mu^*(1)|_p = \mu_p$. If θ is the coordinate on an S^1 or $\mathbb{R}$ orbit, with vector field ∂_θ, then $d\mu = \iota_{\partial_\theta}\omega$
2. Since group is commutative, the (co-)adjoint action is trivial, equivariance becomes invariance: $\mu\left(\phi_g p\right) = \mu\left(p\right)$, in particular $\mathcal{L}_X\mu = 0$.

Example 2.2. Let $G = \mathbb{T}^n$. Now $\mathfrak{g} \simeq \mathfrak{g}^* \simeq \mathbb{R}^n$, and the moment map $\mu : M \to \mathbb{R}^n$, satisfies:

1. Given the standard basis X_i of $\mathfrak{g}$, and $\mu(X_i) = \mu^*(X_i) := \mu_i$, i.e. we have n maps $\mu_i : M \to \mathbb{R}$. If θ_i is the angular coordinate on an S^1 orbit, with vector field ∂_{θ_i}, then $d\mu_i = \iota_{\partial_{\theta_i}}\omega$.
2. Again the group is commutative, so we have G-invariance: $\mu\left(\phi_g p\right) = \mu\left(p\right)$, in particular $\mathcal{L}_{X_i}\mu_j = 0$.

In both of these cases, note that the condition of G invariance implies that the pre-image of a single point in $\mu(M) \subset \mathbb{R}^n$ is contains a G-orbit in M. This observation is key to understanding why these objects are simple to describe, once we know the $\mu(M)$ and how G acts on the pre-image. How to obtain such a description will be the content of the next section. In the meantime, we will describe some examples of Hamiltonian actions, vector fields, symplectic vector fields, and describe $\mu(M)$ in preparation for the convexity theorem.

Example 2.3. Let $G = S^1$, and $M = \mathbb{CP}^1$, be the unit sphere $x^2 + y^2 + z^2 = 1$ centred at the origin in $\mathbb{R}^3$ in cylindrical coordinates (h, θ), where

$$x = h \qquad y = \sin\theta\sqrt{1-h^2} \qquad z = \cos\theta\sqrt{1-h^2}$$

and the symplectic form is given by $\omega = d\theta \wedge dh$. We have that $X_h := \partial_\theta$ is a Hamiltonian vector field, since:

$$\iota_{X_h}\omega = dh$$

Note that as $H^1\left(S^2\right) = 0$, we have that $\Gamma_{\text{Sympl}}(M, \omega) = \Gamma_{\text{Ham}}(M, \omega)$. Now the integral curves ρ_t at $(h, \theta) \in S^2$ are:

$$\rho_t|_{(h,\theta)} = (h, \theta + t)$$

. Thus X_h is the Hamiltonian vector field associated to an S^1-action of rotation around the axis $y = z = 0$, i.e. if $e^{it} \in S^1$ then the action ϕ is given by $\phi_{e^{it}}.(h, \theta) = \rho_t|_{(h,\theta)}$. Then a moment map $\mu : S^2 \to \mathbb{R}$ is given by $\mu = h$. Notice also, that $\mu(S^2) = [-1, 1]$, and $\mu^{-1}((-1, 1)) \cong (-1, 1) \times S^1$, whereas $\mu^{-1}(\{-1\}) \cong \mu^{-1}(\{1\}) \cong \{\text{pt}\}$.

Example 2.4. Let $G = \mathbb{T}^n$, $M = \mathbb{C}^n$ with coordinates $(z_1, \dots, z_n) = \left(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}\right)$, and symplectic form $\omega = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j = \sum_j r_j dr_j \wedge d\theta_j$, the standard Kähler form on $\mathbb{C}^n$.

Now every complete symplectic vector field defines an symplectic action of $\mathbb{R}$ on M via its integral curves ρ_t. If these integral curves are periodic then this descends to a symplectic S^1 action on M. Thus we will look directly for $\mathbb{T}^n$ actions, rather than first defining a vector field. If $t := \left(e^{it_1}, \dots, e^{it_n}\right) \in \mathbb{T}^n$, then:

$$t.\left(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}\right) := \left(r_1 e^{i\theta_1 + t_1}, \dots, r_n e^{i\theta_n + t_n}\right)$$

With vector fields of X_j given by ∂_{θ_j}, then

$$\iota_{\partial_{\theta_j}}\omega = -r_j dr_j = d\left(-\frac{r_j^2}{2}\right)$$

So a moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n$ for this action is given by:

$$\mu(z_1, \dots, z_n) := -\frac{1}{2}\left(|z_1|^2, \dots, |z_n|^2\right)$$

One finds that the image of μ is the negative quadrant, $\mu(\mathbb{C}^n) = \mathbb{R}^n_{\leq 0}$. Moreover, for all $x \in \mathbb{R}^n_{<0}$, $\mu^{-1}(x) \cong \mathbb{T}^n$, while $\mu^{-1}(0) = 0 \in \mathbb{C}^n$.

Example 2.5. Let $G = \mathbb{T}^n$, $M = \mathbb{CP}^n \cong S^{2n+1}/S^1$ with homogeneous coordinates $[z_0 : \dots : z_n]$. We have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & S^1 \arrow[r, hook] & S^{2n+1} \arrow[r, "\pi"] \arrow[d, hook, "i"] & \mathbb{CP}^n \arrow[r] & 0 \\ & & \mathbb{C}^{n+1} & & \end{tikzcd} \tag{6}$$

With $\pi : S^{2n+1} \to \mathbb{CP}^n$ be the quotient map $\pi(z_0, \dots, z_n) = [z_0 : \dots : z_n]$, and $i : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ be the inclusion map. Let $\mathbb{CP}^n$ be equipped with symplectic form defined by the Fubini-Study Kähler metric: the unique symplectic form ω such that the standard Kähler form ω_0 on $\mathbb{C}^n$, given in the previous example, satisfies $i^*\omega_0 = \pi^*\omega$[1].

Now, let $(t_1, \dots, t_n) = t \in \mathbb{T}^n$ act on $\mathbb{CP}^n$ via:

$$t.[z_0 : \dots : z_n] := [z_0 : t_1 z_1 : \dots : t_n z_n]$$

Let $\|z\|^2 = \sum_{j=0}|z_j|^2$, then a moment map for this action is given by:

$$\mu[z_0 : \dots : z_n] := -\frac{1}{2\|z\|^2}\left(|z_1|^2, \dots, |z_n|^2\right)$$

The image of $\mathbb{CP}^n$ under the moment map is given by the polytope:

$$\mu(\mathbb{CP}^n) = \{(x_1 \dots x_n) \in \mathbb{R}^n \mid x_j \leq 0, \sum_j x_j \geq -\frac{1}{2}\}$$

3 Convexity

We will now discuss a result about moment maps concerning $G = \mathbb{T}^n$, where these maps have particularly nice images. This is result is the (abelian) convexity theorem of Atiyah [7] and Guillemin-Sternberg [8]. The theorem itself can be used to prove interesting results, for example in the case, $\dim M = 2n$, and an effective action, they can lead to a classification of manifolds with these actions via polytopes (the higher-dimensional analogue of polygons), done by Delzant in [9]. This is another motivation to study the moment map, as this classification can simplify many calculations and constructions of examples. We prove some preliminary results in the warm-up to convexity theorem.

Lemma 3.1. *Let (M, ω) be a connected compact symplectic manifold, with a symplectic $\mathbb{T}^n$-action, $\phi : \mathbb{T}^n \to \mathrm{Sympl}(M, \omega)$, then there exists an almost complex structure J on M such that $\omega(\cdot, J\cdot)_p$ is a Riemannian metric, and $\phi_\theta^* J = J\, \forall \theta \in \mathbb{T}^n$.*

[1] This happens to be an example of the symplectic quotient construction for a similar moment map to the previous exercise with $S^1 \subset \mathbb{T}^n$.

Proof. It is a standard fact[2], that given a symplectic form can always find a Riemannian metric and an compatible almost complex structure, and let such a metric be given by g_0. Now we have to construct a metric so the second part of the lemma holds, using the Haar measure on $\mathbb{T}^n$, define:

$$g(X,Y) = \int_{\theta\in\mathbb{T}^n} \phi_\theta g_0(X,Y)d\theta$$

Thus $\phi_\theta^* g = g$, and for all $X, Y \in \Gamma(M)$:

$$\begin{aligned} g(X,Y) &= \omega(X,JY) = \phi_\theta^*(X,JY) = \omega(d\phi_\theta X, d\phi_\theta JY) \\ &= \phi_\theta^* g(X,Y) = g(d\phi_\theta X, d\phi_\theta Y) = \omega(d\phi_\theta X, J d\phi_\theta Y) \end{aligned}$$

Thus $Jd\phi_\theta = d\phi_\theta J$ and the claim is proved. $\square$

We use this to say something about the set of fixed points in M of any subgroup $H \subset \mathbb{T}^n$:

Lemma 3.2. *Let (M,ω), ϕ be as before, and $H \subset \mathbb{T}^n$ be a subgroup. Then*

$$\mathrm{Fix}\,(\phi(H)) = \bigcap_{\theta\in H} \mathrm{Fix}\,(\phi(\theta))$$

is a symplectic submanifold of M.

Proof. Let $p \in \mathrm{Fix}\,(\phi(H))$. Since $\mathbb{T}^n$ acts isometrically, then by definition, for the exponential map $\exp : T_pM \to M$, we have:

$$\exp\,(d\phi_\theta X) = \phi_\theta\,(\exp\,(X))$$

Then locally, a fixed point of ϕ_θ is the image under the exponential map of eigenvectors $X \in T_pM$ of $d\phi_\theta$ with eigenvalue 1: i.e. we can construct the local chart of the fixed point set by finding the eigenspace at a point spanned by these vectors, and use the exponential map. In other words:

$$T_p\mathrm{Fix}\,(\phi(H)) = \bigcap_{\theta\in H} \ker\,(I_d - d\phi_\theta)$$

By the implicit function theorem, this is a submanifold, also since $Jd\phi_\theta = d\phi_\theta J$, then the $+1$-eigenspace is preserved by J, so $T_p\mathrm{Fix}\,(\phi(H))$ is also a symplectic vector space with $\omega_p = g\,(X, JY)$. $\square$

The final lemma to prove involves some Morse theory. First, recall the definition:

Definition 3.1. Let (M,g) be a Riemannain manifold, and $f \in C^\infty(M)$ then the **shape operator** $S_f \in C^\infty(M, TM \otimes T^*M)$ is defined by:

$$\mathrm{Hess}f\,(X,Y) = g\,(S_f X, Y)$$

For all $X, Y \in \Gamma(M)$.

Now we define the following class of functions:

Definition 3.2. Let (M,g) be a compact Riemannain manifold, then $f \in C^\infty(M)$ is **Morse-Bott** if:

$$\mathrm{Crit}(f) := \{p \in M \mid df(p) = 0\} \subseteq M$$

is a submanifold, and $\forall p \in \mathrm{Crit}(f)$, $T_p\mathrm{Crit}(f) = \ker S_f(p)$

Furthermore, if f is Morse-Bott, then $\mathrm{Crit}(f)$ has finitely many connected components, thus $\forall p \in \mathrm{Crit}(f)$, we have $T_pM = T_p\mathrm{Crit}(f) \oplus E_p^+ \oplus E_p^-$, where $E_p^\pm$ is the positive/negative-definite eigenspaces of S_f. Note that these subspaces are sub-bundles of TM restricted to each component of $\mathrm{Crit}(f)$, and we denote the rank of these subbundles as the (co-)index. We will use the following from [12, p.178-179]:

Lemma 3.3. *Let (M,g) be a compact connect Riemannian manifold, and $f \in C^\infty(M)$ be Morse-Bott function such for every connected component of* $\mathrm{Crit}(f)$*, the index and co-index is not equal to one. Then $f^{-1}(c)$ is connected for every $c \in \mathbb{R}$.*

Now we return to our situation:

[2] e.g. see [11, App. 17.1]

Lemma 3.4. *Let (M, ω) be a connected compact symplectic manifold, with a Hamiltonian $\mathbb{T}^n$-action, $\phi : \mathbb{T}^n \to \mathrm{Sympl}(M, \omega)$, and moment map $\mu : M \to \mathbb{R}^n$. Let $X \in \mathbb{R}^n$, and $\mu^*(X) \in C^\infty(M)$, and $\mathbb{T}^n \subseteq H := \overline{\{\exp tX \mid t \in \mathbb{R}\}}$ be the closure of the the subgroup generated by X. Then:*

1. $\mathrm{Crit}\,(\mu^*(X)) = \bigcap_{\theta \in H} \mathrm{Fix}\,(\phi(\theta))$
2. *$\mu^*(X)$ is Morse-Bott, and has criticial submanifolds of even index/co-index*

Proof. $d\mu^*(X)|_p = \iota_{D\phi(X)}\omega|_p$, then if $p \in \mathrm{Crit}\,(\mu^*(X))$ we have that $D\phi(X)|_p = 0$, and so the flows generated by this action leave x invariant, thus $p \in \bigcap_{\theta \in H} \mathrm{Fix}\,(\phi(\theta))$. Similarly if we assume the converse, then $D\phi(X)|_p = 0$, and if we have that X is irrational so that $H = G$, we have converse inclusion. By continuity we may extend this to rational X.

We now show $T_p\mathrm{Crit}(d\mu^*(X)) = \ker S_{\mu^*(X)}(p) = \bigcap_{\theta \in H} \ker\,(I_d - d\phi_\theta)$. Note that $d\psi_{\exp tX} = \exp\left(-tJS_{\mu^*(X)}\right)$, so if X is irrational, then $kerS_{\mu^*(X)}$ are the fixed point of the action, and by continuity we prove the claim for non-rational X.

Finally, we saw that $Jd\phi_\theta = d\phi_\theta J$. Since the critical submanifolds must be even dimensional as they are symplectic, and J is almost complex, thus the negative/positive eigen-spaces must have even dimension as well. $\square$

Theorem 3.5 (Convexity, [7] [8])**.** *Let (M, ω) be a connected compact symplectic manifold, $\mu : M \to \mathbb{R}^n$ be the moment map for a Hamiltonian $\mathbb{T}^n$-action, with $\phi : \mathbb{T}^n \to \mathrm{Sympl}(M, \omega)$. Then:*

1. *$\mu^{-1}(c)$ is connected $\forall c \in \mu(M)$.*
2. *$\mu(M)$ is convex.*
3. *$\mu(M) = \mathrm{co}\,(\mu(\mathrm{Fix}(\phi)))$, the convex hull of the images of the fixed points of the action.*

Proof. As for the first two statements, we will prove this via the following strategy using induction on $1 \le k \le n$. Let the statement of the theorem for all $\mathbb{T}^k \dots \mathbb{T}^1$ be:

- A_k : $\mu^{-1}(c)$ is connected $\forall c \in \mu(M)$.
- B_k : $\mu(M)$ is convex.

B_1 is easy to prove: since M is connected, and the map $\mu : M \to \mathbb{R}$ is continuous, then $\mu(M) \subset \mathbb{R}$ is connected. Hence, it must also be convex. Next, we show $A_k \Rightarrow B_{k+1}$. Now since this is a torus action we can split the moment map: $\mu = (mu_1, \dots \mu_k, \mu_{k+1})$. Let us first assume we have a line of integer slope, so that we split $\mathbb{R}^{n+1} = L \oplus P$, and P is a plane with normal of integer slope containing 0, so that P generates a sub-torus action $\mathbb{T}^l \subset \mathbb{T}^{k+1}, l \le k$, with map $\mu_P := \mathrm{proj}_P\mu$. For $v_0 \in \mu_P(M)$ be in P, where L is some line not in P, so that if we fix $p_0 \in \mu_P^{-1}(v_0)$, then $\mu_P^{-1}(v_0) = \{p \in M \mid \mu(p) - \mu(p_0) \in L\}$. Since $\mu_P^{-1}(v_0)$ is connected by assumption, and if p_t is a path connecting p_0 and any other point $p_1 \in \mu_P^{-1}(v_0)$, then the path $\mu(p_t) - \mu(p_0) \in L$. But L is one dimensional, so connectivity implies convexity, thus for any $p_1, p_0 \in \mu_P^{-1}(v_0)$, the line containing $\mu(p_1), \mu(p_0)$ is in $\mu(M)$, i.e. $t\mu(p_0) + (1-t)\mu(p_1) \in \mu(M)$. Since M is compact, $\mu(M)$ is also compact in $\mathbb{R}^n$, so closed, thus the line connecting any points $p_0, p_1 \in \mu(M)$ can be approximated by an integer line sufficiently close.

As for the third part of the theorem, note that the second statement in the theorem implies inclusion $\mu(M) \subseteq \mathrm{co}\,(\mu(\mathrm{Fix}(\phi)))$, as by definition the convex hull of a set is the smallest convex set containing it. Also, we note that the moment map is constant on $\mathrm{Fix}\,(\phi)$, so the image of the fixed points in M is a set of points in $\mu(M)$. Denote this set $\{v_1 \dots v_N\}$, and let $\langle \cdot, \cdot \rangle$ be the pairing between $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. Since $\mathrm{co}\,(\{v_1 \dots v_N\})$ is a convex subset of $\mathbb{R}^n$, for every $v_0 \notin \mathrm{co}\,(\{v_1 \dots v_N\})$ there exists $X \in \mathfrak{g}$ such that:

$$\langle v_0, X \rangle > \langle v_i, X \rangle$$

For all $1 \le i \le N$, in particular, one may choose X such that X has irrational slope, so that $\exp tX$ is dense in $\mathbb{T}^n$. Since $d\mu^*(X) = \iota_{D\phi(X)}\omega$ then $D\phi(X)|_p = 0 \Leftrightarrow p \in \mathrm{Fix}(\phi)$. Furthermore, since M is compact, this implies that $\mu^*(X) \in C^\infty(M)$ must attain a maximum on $p \in \mathrm{Fix}(\phi)$, but now:

$$\langle v_0, X \rangle > \sup_{p \in M} \langle \mu(p), X \rangle$$

Hence $v_0 \notin \mu(M)$.

Now we turn to A_1 and the inductive step $A_k \Rightarrow A_{k+1}$. For A_1, this is a direct consequence of (3.3) applied to the Morse function $\mu^*(X)$. It suffices to prove the inductive step in the case we have an effective action i.e. $(d\mu_1 \dots d\mu_k)$ are linearly independent, otherwise we may reduce to the action of a sub-torus on which the action is effective. Now by this assumption, let $0 \neq X \in \mathfrak{g}$, then:

$$d\mu^*(X) = \sum_i X_i d\mu_i = 0 \Leftrightarrow X_i = 0 \forall i$$

Thus $d\mu^*(X)$ is non-constant. Now consider:

$$\mathcal{C} := \bigcup_{X \in \mathbb{R}^n \neq 0} \text{Crit}\,(\mu^*(X)) = \bigcap_{X \in \mathbb{Z}^n \neq 0} \text{Crit}\,(\mu^*(X))$$

Now $\mathcal{C}$ is a countable union of proper submanifolds, and by Baire's theorem, this implies $M - \mathcal{C}$ is dense in M. Also note that $M - \mathcal{C}$ is open since $p \in M - \mathcal{C} \Leftrightarrow d\mu^*(X)(p) = \sum_i X_i d\mu_i(p) \neq 0$ for all $X \in \mathbb{R}^n$, i.e. $d\mu_i(p)$ are linearly independent, thus by continuity they must also be linearly independent in a neighbourhood of p. This shows that the regular values of μ is dense in $\mu(M)$. Then again, by continuity, to show $\mu^{-1}(v)$ is connected for every $v = (v_1, \dots, v_k) \in \mu(M)$, it suffices to show that it is connected for the regular values of μ. Furthermore it suffices to show this for $(v_1, \dots, v_{k-1})$ and $(\mu_1, \dots, \mu_{k-1})$. Now by the inductive hypothesis:

$$Q = \bigcap_{j=1}^{k-1} \mu_j^{-1}(v_j)$$

is connected whenever $(v_1, \dots, v_{k-1})$ is a regular value for $(\mu_1, \dots, \mu_{k-1})$. Then, using a lemma from [12, p.183], and lemma (3.4), then $\mu^{-1}(v) = Q \cap \mu_j^{-1}(v_j)$ is connected for every $j \leq k-1$, thus the claim is proved. $\square$

References

[1] M. Gromov, "Pseudo holomorphic curves in symplectic manifolds," *Inventiones mathematicae*, vol. 82, pp. 307–347, Jun 1985.

[2] J. Sparks, "Sasaki-Einstein manifolds," in *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics*, vol. 16 of *Surv. Differ. Geom.*, pp. 265–324, Int. Press, Somerville, MA, 2011.

[3] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, "Hyper-Kähler metrics and supersymmetry," *Comm. Math. Phys.*, vol. 108, no. 4, pp. 535–589, 1987.

[4] T. B. Madsen and A. Swann, "Multi-moment maps," *Adv. Math.*, vol. 229, no. 4, pp. 2287–2309, 2012.

[5] D. Martelli and J. Sparks, "Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals," *Comm. Math. Phys.*, vol. 262, no. 1, pp. 51–89, 2006.

[6] D. Cox, J. Little, and H. Schenck, *Toric Varieties.* Graduate studies in mathematics, American Mathematical Society, 2011.

[7] M. Atiyah, *Michael Atiyah collected works. Vol. 5: Gauge theories.* 1988.

[8] V. Guillemin and S. Sternberg, "Convexity properties of the moment mapping," *Inventiones mathematicae*, vol. 67, pp. 491–513, Oct 1982.

[9] T. Delzant, "Hamiltoniens p'eriodiques et images convexes de l"application moment," *Bull. Soc. Math. Fr.*, vol. 116, 01 1988.

[10] A. da Silva, *Lectures on Symplectic Geometry.* Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2004.

[11] C. H. Taubes, *Differential Geometry: Bundles, Connections, Metrics, and Curvature.* No. 23 in Oxford Graduate Texts in Mathematics, Oxford University Press, 1st ed., 2011.

[12] D. McDuff and D. Salamon, *Introduction to Symplectic Topology.* Oxford Mathematical Monographs, Oxford University Press, 2017.