Nearly Kähler 6-Manifolds

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Abstract

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1 Motivation

A useful way to a way to construct manifolds with special holonomy is via Riemannian cones:

Definition 1.1. Given a Riemannian manifold (M, g), the *cone over* M, denoted $\mathcal{C}(M)$, is defined as the space $\mathbb{R}_{>0} \times M$ with the metric $dr^2 + r^2 g$.

For example, (M, g) has a Sasaki-Einstein structure if and only if $\mathcal{C}(M)$ is Calabi-Yau. Recall that if $\mathcal{C}(M)$ is Ricci-flat then g is Einstein, and if $\mathcal{C}(M)$ is Kähler, then (M, g) has a Sasaki structure. There are infinitely many manifolds with Sasaki-Einstein structures, so there are infinitely many Calabi-Yau cones [1].

On the other hand, if we want to study desingularization or deformation of special holonomy manifolds, cone singularities are a particularly simple type of singularity to start with. We can, for example, de-singularize cones by replacing the singular point at $r = 0$ with another manifold, to get an asymptotically conical manifold near the singular point. We can approximate the metric of the cone to arbitrary precision by the asymptotically conical metric away from the cone point.

2 Four Definitions

2.1 Motivating definition: Holonomy in G_2

In this section we give four equivalent definitions of strictly (i.e. non-Kähler) nearly-Kähler six manifolds. All references for this section can be found in [2]. First some preliminary definitions:

Definition 2.1. Let (M, g) be a Riemannian 6-manifold. An SU(3) structure on (M, g) is a pair (Ω, ω), where ω is a non-degenerate 2-form compatible with g and Ω is a non-vanishing $(3, 0)$ form with respect to the almost-complex structure induced by g and ω, so that up to normalisation:

$$\omega^3 = \Omega \wedge \bar{\Omega}$$

Let (M, g) be a Riemannian six-manifold with an SU(3) structure (Ω, ω).

Definition 2.2. (M, g) is a nearly Kähler 6-manifold if the holonomy of $\mathcal{C}(M)$ is contained in G_2.

Remark 2.3. In fact, if (M, g) is complete and $\mathrm{Hol}(\mathcal{C}(M)) \subsetneq G_2$, then $\mathcal{C}(M) = (\mathbb{R}^7, g_0)$, where g_0 is the Euclidean metric on $\mathbb{R}^7$, i.e. $M = S^6$.

Note also that having a G_2-structure on the cone gives a natural $\mathrm{SU}(3) \subset G_2$ structure on the level set $1 \times M$. We will see this later on. An explicit construction of the G_2-structure starting from the SU(3)-structure is given in [3].

2.2 Simply-connected, complete examples

Four examples come from homogeneous spaces and were discovered by Gray, Wolf in [4]. A result of Butruille in [5] is that these are the only simply-connected examples of nearly Kähler 6-manifolds coming from homogeneous spaces:

1. $(S^6, g_0) \simeq G_2/\mathrm{SU}(3)$.

2. $S^3 \times S^3 \simeq SU(2)^3/\Delta SU(2)$

3. $\mathbb{C}P^3 \simeq Sp(2)/U(1) \times Sp(1)$

4. $F_3 \simeq SU(3)/T^2$.

Two more non-homogeneous examples are the so-called co-homogeneity one spaces. These were discovered by Foscolo and Haskins in 2017 [2]. Topologically, they are given as:

1. S^6

2. $S^3 \times S^3$

Note that these spaces do not carry their homogeneous metrics. This is a complete list of all the current known examples of complete, simply-connected nearly-Kähler 6-manifolds.

Remark 2.4. The first explicit example of a local metric with holonomy equal to G_2 was a cone over the flag manifold F_3 by Bryant in [6]. Bryant and Salamon in [7] constructed the first complete examples of G_2 metrics which were asymptotic to the cones $\mathcal{C}(S^3 \times S^3)$, $\mathcal{C}(\mathbb{C}P^3)$, and $\mathcal{C}(F^3)$.

2.3 Second definition: formulas for $d\omega$ and $d\Omega$

Recall that a G_2 manifold M is equipped with a closed, G_2-invariant 3-form ϕ and its Hodge dual $*\phi$. If M is the cone over a six-manifold with an SU(3) structure (Ω, ω), these are given by the equations:

$$\phi = r^2 dr \wedge \omega + r^3 \operatorname{Re}(\Omega)$$
$$*\phi = -r^3 dr \wedge \operatorname{Im}(\Omega) + \frac{1}{2} r^4 \omega^2$$

Imposing that these forms are closed, then the resulting equations for $d\omega$ and $d\operatorname{Im}(\Omega)$ motivates our second definition of a nearly Kähler six-manifold.

Definition 2.5. Let (M, g) be a Riemannian 6-manifold with an SU(3) structure (Ω, ω). Then (M, g) is a nearly Kähler six-manifold if:

$$d\omega = 3\operatorname{Re}(\Omega) \qquad \text{and} \qquad d\operatorname{Im}(\Omega) = -2\omega^2.$$

For more on this, see [2].

2.4 Facts about nearly Kähler six-manifolds

The following facts follow easily from our above discussion and the first definition of a nearly Kähler six-manifold.

- If $\mathcal{C}(M)$ is Ricci flat then (M, g) is Einstein, so it has constant (positive) scalar curvature. This implies that if (M, g) is complete, then M is compact and $\pi_1(M)$ is finite.
- Since $\Omega \in \Lambda^{3,0}(M)$ is nonvanishing, we have that:

$$c_1(M, \omega) := c_1(\Lambda^{3,0}(M)) = 0.$$

2.5 Third definition: covariant derivative of J

For more details of the following, see [8]. Recall that the octonion ring can be written $\mathbb{O} = \mathbb{H} \oplus K\mathbb{H}$ where $\mathbb{H}$ is the quaternion ring and K is a formal variable such that $K^2 = -1$. The algebra on $\mathbb{O}$ is determined by the relations

$$\overline{q_1 + Kq_2} = \overline{q_1} - Kq_2$$
$$(q_1 + Kq_2).(q_3 + Kq_4) = q_1 q_3 + \overline{q_4} q_2 + K(q_2 \overline{q_3} + q_4 q_1).$$

The imaginary octonions are the elements $x \in \mathbb{O}$ of the form $ai + bj + ck + K(x + iy + jz + kw)$. Octonion multiplication induces a cross product on the imaginary octonions by the map:

$$X \times Y = \frac{1}{2}(X.Y - Y.X)$$

for X and Y imaginary octonions.

Now consider the example (S^6, g_0). S^6 is embedded in $\mathbb{R}^7$, which can be identified with the imaginary octonions and given the resulting cross product structure. If N is the normal vector field to S^6, then

$$J = N \times X$$

gives an almost-complex structure on S^6. It can be checked that the Nijenhuis tensor of (S^6, J, g_0) is nonvanishing, and that J is almost-Hermitian with respect to g_0. Define a 2-form on S^6 by

$$\omega(X, Y) = \langle N \times X, Y \rangle,$$

where the angle brackets denote the Euclidean inner product. Note that this is form invariant under the action of G_2 on $\mathrm{Im}(\mathbb{O})$, and one can use this form to recover a 3-form Ω, to give the $SU(3)$ structure on the homogeneous space $(S^6, g_0) \simeq G_2/\mathrm{SU}(3)$.

Remark 2.6. This cross product also induces a standard G_2 3-form on $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ via:

$$\phi(X, Y, Z) = \langle X \times Y, Z \rangle$$

Lemma 2.1. *On (S^6, J, g_0), we have that*

$$\langle (\nabla_X J)Y, Z \rangle = \langle \nabla_X N \times Y, Z \rangle,$$

and so $(\nabla_X J)X = 0$, but $\nabla J \neq 0$.

Recall also that if $\nabla J = 0$, then $d\omega = 0$. This motivates our third definition.

Definition 2.7. Let J be an almost-Hermitian structure on Riemannian manifold (M, g). Then M is nearly Kähler if $(\nabla_X J)X = 0$ $\forall X \in \Gamma(TM)$ but $\nabla J \neq 0$, where ∇ is the Levi-Civita connection.

We may translate this observation/definition into representation theory. We begin point-wise: let V be a real $2n$ dimensional vector space with inner product g and ω an almost-Hermitian structure. Then g induces a natural inner product on $\otimes^3 V^*$, and $\mathrm{U}(n)$ acts on $\otimes^3 V^*$ in the natural way.

Theorem 2.2 (Gray-Hervella [9])**.** . *Let $W \subset \otimes^3 V^*$ be the subspace*

$$W = \{a \in \otimes^3 V^* : a(X, Y, Z) = -a(X, Z, Y) = -a(X, JY, JZ) \quad \forall X, Y, Z \in V\}.$$

Then W splits orthogonally into irreducible $U(n)$-representations

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where

$$W_1 = \{a \in W : \alpha(X, X, Z) = 0\}.$$

For $n = 1$, $W = 0$. For $n = 2$, $W_1 = W_3 = 0$. For $n \geq 3$, all W_i are nontrivial.

Now take $V = T_pM$, W the subspace of $\otimes^3 V^*$ as before, and $\alpha = (\nabla\omega)_p$. Then we have the following fact.

Lemma 2.3. *The form $\alpha \in W$ for all $p \in M$ if and only if (M, g) is almost-Hermitian. Furthermore, if we assume $\nabla J \neq 0$, then (M, g) is nearly-Kähler if and only if $\alpha \in W_1$ for all $p \in M$.*

Therefore the first dimension at which we get strictly nearly-Kähler manifolds is 6 real dimensions (3 complex dimensions). Note that definition 2.7 makes sense in all even dimensions, but we see nearly-Kähler 6-manifolds play a special role.

2.6 Fourth definition: real nonzero Killing spinor

Recall from the previous talk that $\mathcal{C}l(6, 0) \simeq \mathrm{End}(\mathbb{R}^8)$. The group Spin(6) acts on the spin representation $\Delta \simeq \mathbb{R}^8$. If we fix an orthonormal basis $e_1, ..., e_6$ for the $\mathbb{R}^6$ on which we have formed $\mathcal{C}l(6, 0)$, then the e_i's generate $\mathcal{C}l(6, 0)$ as an algebra.

Let $J = e_1 \cdot ... \cdot e_6$, where $\cdot$ denotes Clifford multiplication. Then for any $0 \neq \phi \in \Delta$, Δ can be decomposed as

$$\Delta = \mathbb{R}\phi \oplus \mathbb{R}J\phi \oplus \{x \cdot \phi : x \in \mathbb{R}^6\}.$$

Multiplication by J fixes the last (six-dimensional) component. Thus we can define a complex structure J_ϕ on $\mathbb{R}^6$ by the equation

$$J_\phi(x).\phi = J(x \cdot \phi). \tag{1}$$

We can also define the form Ω_ϕ by

$$\Omega_\phi(X, Y, Z) = -\langle X \cdot Y \cdot Z \cdot \phi, \phi\rangle_{\mathbb{R}^8}.$$

It is checked that J_ϕ and Ω_ϕ give an SU(3) structure on $\mathbb{R}^6$. Now let ϕ be a section of the spin bundle $S = \tilde{P} \times_{\text{Spin}} \Delta$, and x be a vector field on M: then this point-wise construction gives an SU(3) structure on S.

Lift the Levi-Civita connection on M to the connection ∇^S on S. We call a section $\phi \in \Gamma(S)$ a real Killing spinor if for all $X \in \Gamma(TM)$,

$$\nabla^S_X(\phi) = \lambda X \cdot \phi$$

for some nonzero $\lambda \in \mathbb{R}$. By differentiating equation 1, we can show that if M admits a real Killing spinor then it satisfies definition 2.7 with the SU(3) structure defined above. See [10] for a complete proof.

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